Bachelor of Science
(B.Sc.- PCM)

PARTIAL DIFFERENTIATION EQUATIONS
(DBSPDS303T24)

Self-Learning Material
(SEM-III)

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COURSE INTRODUCTION

A function's partial derivative (PD) is its derivation with regard to one variable while maintaining constant values of other variables. The steps involved in computing a partial derivative are identical to those involved in computing an ordinary derivative, with the exception that the variables we are differentiating with regard to the other variables are treated as constants. The course is divided into 10 units. Each Unit is divided into sub topics. The Units provide students with a comprehensive understanding of the real number system and its characteristics. They also examine continuity, differentiability, and integrability concepts in a rigorous mathematical framework, analyze sequences and series of real numbers and functions, and apply these concepts to solve theoretical and practical problems.

There are sections and sub-sections inside each unit. Each unit starts with a statement of objectives that outlines the goals we hope you will accomplish. Every segment of the unit has many tasks that you need to complete.

We wish you pleasure in the Course. Please try all of the exercises and activities included in the units. We are certain that you will get better at math if you follow through on it.

Course Outcomes: After the completion of the course, the students will be able to:

1. Recall the Understanding of PDE Basics.
2. Explain Solution Techniques of PDE.
3. Understand and apply methods for solving non-linear PDEs in special cases.
4. Analyze and Implement numerical methods for solving PDEs, including finite difference methods, finite element methods, and other discretization techniques.
5. Utilize software tools and programming languages to implement and solve PDEs numerically uniform continuity, differentiation, integration and uniform convergence.
6. Create a theoretical understanding of the existence, uniqueness, and regularity of solutions to PDEs.

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UNIT - 1

Introduction of Partial Derivates

Learning Objectives

- Differentiate functions of multiple variables with respect to one variable while treating other variables as constants.
- Identify and analyze real-world applications of partial derivatives, such as optimization problems in economics, physics, and engineering.
- Introduce second and higher-order partial derivatives and their interpretations in terms of curvature and mixed derivatives.

Structure

1.1 Partial Derivative

A function's partial derivative (PD) is its derivation with regard to one variable while maintaining constant values of other variables. The steps involved in computing a partial derivative are identical to those involved in computing an ordinary derivative, with the exception that the variables we are differentiating with regard to the other variables are treated as constants.

Let's study partial derivative calculation with examples for various orders in more detail.

What is Partial Derivative?

A partial derivative is a concept from calculus that extends the idea of ordinary derivatives to functions of more than one variable. It measures how a function changes with respect to one of its variables while holding the other variables constant.

The Partial Derivative of multivariable function with respect to one variable, when other variables are as constants. Let z = f(x, y), then derivative with respect to one of the variables x or y. As an illustration,

- If y taken as unchanged to determining partial derivative of f(x, y) with regards to x (denoted by ∂f / ∂x).
- If x taken as unchanged in order to calculate partial derivative of f(x, y) with regard to y (denoted by ∂f / ∂y).
- The term "partial" comes from the fact that, when performing partial differentiation, we only consider one variable at a time rather than all of the variables as variables. The derivative's limit definition and the limit definition of a partial derivative bear a striking resemblance. By applying the following limit formulas, we may determine the partial derivatives:

If z = f(x, y), then

- $\frac{\partial f}{\partial x} = \lim_{h \to \infty} \frac{[\,f(x+h,\ y) - f(x,\ y)\,]}{h}$
- $\frac{\partial f}{\partial y} = \lim_{h \to \infty} \frac{[\,f(x,\ y+h) - f(x,\ y)\,]}{h}$

These conditions, which apply the first principle, are similar to the derivative definition.

Example 1:

Find ∂f / ∂x, if f (x, y) = xy.

Solution:

$$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{[\, f(x + h,\ y) - f(x,y)\,]}{h}$$

$$= \lim_{h\to 0} \frac{[(x + h)\, y - x\, y]}{h}$$

$$= \lim_{h\to 0} \frac{[xy + hy - x\, y]}{h}$$

$$= \lim_{h\to 0} \frac{[hy]}{h}$$

$$= \lim_{h\to 0} y$$

= y

Hence $\frac{\partial f}{\partial x} = y.$

Partial Derivative Symbol

Now, we aware with notations dy/dx, d/dx (y), d/dx (f(x)), f '(x), etc. are used to represent the ordinary derivative of a function y = f(x). Instead of using "d" to express a partial derivative, we use the notation "∂". "∂" is pronounced "doh" in our language, but it goes by other names as well, such as "partial", "del", "partial dee", "dee", "Jacobi's delta", and so on. Given a function in two variables, z = f(x, y), then

- $\frac{\partial f}{\partial x}$ is f's partial derivative in relation to x.
- $\frac{\partial f}{\partial y}$ is f's partial derivative in relation to y.

Partial derivatives are represented by different notations, in the same way that ordinary derivatives are. Fx, fx', Dxf, ∂ / ∂x (f), ∂x f, ∂ / ∂x [f(x, y)], ∂z / ∂x, and so on can all be represented as ∂f / ∂x.

Calculate Partial Derivatives

As we've just seen, the partial derivatives are found using the limit definitions. However, calculating the limit and using the limit formula are not always simple tasks. This means that, just from its definition, we have another way to compute partial derivatives. The following procedures can be used to calculate the partial derivatives in this technique if the function is $z = f(x, y)$:

- Step 1: Determine the variable that needs to have the partial derivative found.
- Step 2: Treat all other variables as constants, with the exception of the variable from Step 1.
- Step 3: Apply the standard differentiation rules to differentiate the function.

Example 2:

Let's use the previous stages to solve the identical case above: if $f(x, y) = xy$, then determine the partial derivative $\partial f / \partial x$.

Resolution:

We need to determine $\partial f / \partial x$. This means that we need to ascertain f's partial derivative with regard to x. Thus, we handle y as a constant. As a result, 'y' can be written outside the derivative (because, in ordinary differentiation, 'c' is a constant and the rule is $d/dx\,(c\,y) = c\,dy/dx$). Hence, $y\ \partial / \partial x\,(x) = y\,(1)$ and $\partial f / \partial x = \partial / \partial x\,(xy)$ The power rule states that $d/dx\,(x) = 1$) equals y.

1.2 First Order Partial Derivatives

A function in two variables, denoted by $z = f(x, y)$, can have two first-order partial derivatives: $\partial f / \partial x$ and $\partial f / \partial y$.

Example 3:

If $z = x^2 + y^2$, find all the first order partial derivatives.

Solution:

$f_x = \partial f / \partial x = (\partial / \partial x)\,(x^2 + y^2)$

$= (\partial / \partial x)\,(x^2) + (\partial / \partial x)\,(y^2)$

$= 2x + 0$ (as y is a constant)

$= 2x$

$f_y = \partial f / \partial y = (\partial / \partial y)\,(x^2 + y^2)$

$= (\partial / \partial y)\,(x^2) + (\partial / \partial y)\,(y^2)$

$= 0 + 2y$ (as x is a constant)

$= 2y$

1.3 Second Order Partial Derivatives

Differentiating the function with respect to the stated variables one after the other yields the second-order partial derivative. Four second-order partial derivatives can be found for a function in two variables, z = f(x, y): ∂2f / ∂x2, ∂2f / ∂y2, ∂2f / ∂x ∂y, and ∂2f / ∂y ∂x.

In order to determine them as follows

- $f_{xx} = \partial^2 f / \partial x^2 = \partial / \partial x\ (\partial f / \partial x) = \partial / \partial x\ (f_x)$
- $f_{yy} = \partial^2 f / \partial y^2 = \partial / \partial y\ (\partial f / \partial y) = \partial / \partial x\ (f_y)$
- $f_{yx} = \partial^2 f / \partial x\ \partial y = \partial / \partial x\ (\partial f / \partial y) = \partial / \partial x\ (f_y)$
- $f_{xy} = \partial^2 f / \partial y\ \partial x = \partial / \partial y\ (\partial f / \partial x) = \partial / \partial y\ (f_x)$

Take note of the notations fxy and fyx. The partial differentiation order is shown by the variables' order in each subscript. For instance, fyx, which is the same as −2f / ∂x ∂y, denotes partial differentiation first with regard to y and then with respect to x.

Example 4:

Find 2nd order PD, If $z = x^2 + y^2$.

Solution:

Let $z = f(x, y) = x^2 + y^2$.

Then $f_x = 2x$ and $f_y = 2y$.

Now, $f_{xx} = \partial / \partial x(f_x) = \partial / \partial x(2x) = 2$

$f_{yy} = \partial / \partial y(f_y) = \partial / \partial y(2y) = 2$

$f_{yx} = \partial / \partial x(f_y) = \partial / \partial x(2y) = 0$

$f_{xy} = \partial / \partial y(f_x) = \partial / \partial y(2x) = 0$

Hence $f_{yx} = f_{xy}$.

We conclude that the order of PD is equal.

Partial Differentiation Formulas

Partial differentiation is the method of determining partial derivatives. The following limit formulas are used to derive the first-order partial derivatives (as previously discussed) of a function z = f(x, y):

- $\lim h \to 0$ [f(x + h, y) - f(x, y)] / h = ∂f / ∂x

- $\lim h \to 0$ [f(x, y + h) - f(x, y)] / h = ∂f / ∂y

However, the process of partial differentiation would be much simpler if we just treated all the variables other than the variable we are partially differentiating as constants rather than employing these formulas. We just apply the standard differentiation rules in this procedure; among them, the following are the key rules:

1.4 Power Rule

In differentiation, the power rule states that (d/dx0(xn) = nxn-1. Partial derivatives can also be handled using the same principle.

Example 5:

$\partial / \partial x (x^2y)$=?

Solution:

$\partial / \partial x (x^2y) = y \, \partial / \partial x (x^2) = y (2x) = 2xy$.

1.5 Product Rule

Ordinary differentiation's product rule states that d/dx (uv) = u dv/dx + v du/dx. When there are two functions for a single variable, we may also use the same principle in partial differentiation.

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Example 6:

∂/∂x (xy sin x) =?

Solution:

∂/∂x (xy sin x) = y ∂ / ∂x (x / sin x)

= y [(sin x ∂ / ∂x (x) + x ∂ / ∂x (sin x))]

= y [sin x + x cos x]

1.6 Rule of Quotient
Ordinary differentiation's quotient rule gives d/dx (u/v) = [v du/dx - u dv/dx] / v2. This rule can be used, along with other rules, to find PD.

Example 7:

$\partial/\partial x\ (xy / \sin x) = ?$ $\partial/\partial x\ (xy \sin x) = ?$

Solution:

$\partial / \partial x\ (xy / \sin x)$

$= y\, \partial / \partial x\ (x / \sin x)$

$= y\, [\ (\sin x\ \partial / \partial x\ (x) - x\ \partial / \partial x\ (\sin x)\ ) / \sin^2 x\ ]$

$= y\, [\ \sin x - x \cos x] / \sin^2 x$

1.7 Chain`s Rule

Compared to conventional derivatives, the chain rule of partial derivatives operates somewhat differently. The rule can occasionally apply to both ordinary and partial derivatives. This rule has multiple applications, and which one to use depends on how the function's variables are as

- If y = f(x) is a function where x = x(u,v),
- If z = f(x, y), where each of x = x(t) and y = y(t) then

$$df/dt = (\partial f/\partial x \cdot dx/dt) + (\partial f/\partial y \cdot dy/dt)$$

- If x = x(u, v) and y = y(u, v), and z = f(x, y), then

$$\partial f/\partial u = \partial f/\partial x \cdot \partial x/\partial u + \partial f/\partial y \cdot \partial y/\partial u$$

$$\partial f/\partial v = \partial f/\partial x \cdot \partial x/\partial v + \partial f/\partial y \cdot \partial y/\partial v$$

Example 8:

Find $\partial f/\partial u$, if $z = e^{xy}$, where x = uv and y = u + v.

Solution:

Using chain rule

$\partial f/\partial u = \partial f/\partial x \cdot \partial x/\partial u + \partial f/\partial y \cdot \partial y/\partial u$

$= \partial / \partial x\ (e^{xy}) \cdot \partial / \partial u\ (uv) + \partial / \partial y\ (e^{xy}) \cdot \partial / \partial u\ (u + v)$

$= (e^{xy} \cdot y)\ (v) + (e^{xy} \cdot x)\ (1)$

$= e^{xy} (x + vy)$

Other Rules of Partial Differentiation

- If f(x, y) = a constant, then dy/dx =-f_x/f_y.
- For any two functions u(x, y) and v(x, y), the determinant
  |(∂u/∂x)(∂u/∂y)(∂v/∂x)(∂v/∂y)||||(∂u/∂x)(∂u/∂y)(∂v/∂x)(∂v/∂y)|
  is known as **Jacobian** of u and v.
- If f(x, y, z) is a function in three variables. The Laplace equation for partial derivatives is ∂2f/∂x2 + ∂2f/∂y2 + ∂2f/∂z2 = 0. The harmonic function is any function f that fulfils the Laplace equation.

1.8 Maxima and Minima

We refer to Maxima and Minima as the function's crucial points. In any function, a minima is a low point and a maximum is a high point. There may be more than one maximum and minimum point in a function. Maxima and minima are the places at which the function achieves its maximum and minimum values.

Relative Maxima and Minima

The positions at which the function provides the largest and least values, respectively, in their respective neighborhoods are known as relative maxima and minima. Any function can readily find its relative maxima and minima by applying the first derivatives and second derivative tests, respectively. The graph that is superimposed upon a function's neighborhood's relative maxima and minima (Figure 4.8.1)

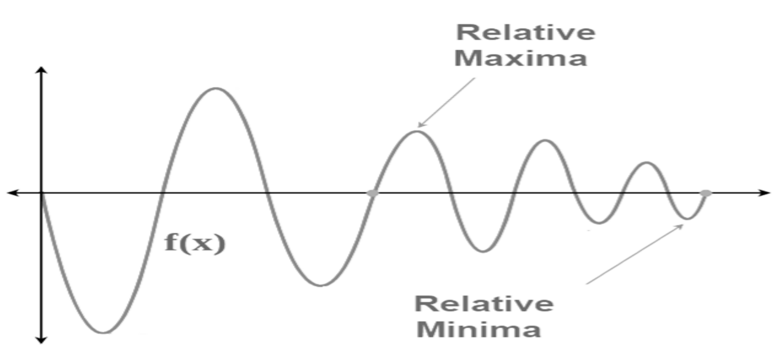

Figure 1.1 Relative Maxima & Minima

Points of Maxima and Minima (First Derivative Test)

The points at which a smoothly moving function flattens out are known as minima or maxima. Two questions now arise in response to this remark.

1. How can I identify the locations where a function flattens out?
2. Assume we have a critical point—the point at which the function flattens. How can I determine if it's a maximum or a minimum?

Let's examine the function's slope in order to respond to the first query. Zero slopes are found at the places when the function flattens out. We are aware that the derivative is only the function's slope at a specific point in time. Thus, our goal is to locate the derivative's zero points. Consequently, the First Derivative Test is another name for this test. Next, we

$f'(x) = 0$

The critical points' locations are provided by the equation's solution. These key points identify the locations where the curve's tangent is parallel to the x-axis, but we are still unsure of whether these are the places of maxima or minima for the purposes of the Second Derivative test.

Recognizing Maxima and Minima

The derivative is at its maximum if its sign is positive prior to the critical point and negative subsequent to it, as the picture below illustrates. In a similar vein, if it is positive after the key moment and negative prior to it. It is the barest minimum. The second derivative test can also be used to identify maxima and minima.

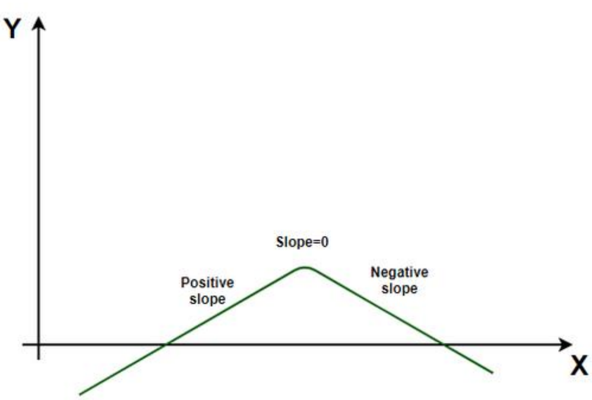

Figure 1.2 Maxima & Minima Slope

Second Derivative Test

The second derivative, f''(x) at that point, is utilized to determine whether we have maxima or minima when the slope of a function is zero at x.

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1. If f'’(x) < 0, then maxima.
2. If f'’(x) > 0, then minima.

Note: The test is deemed unsuccessful if, at the key locations, the second derivative is zero.

Steps to Find Relative Maxima and Minima

The procedures shown below can be used to discover the relative maxima and minima of a function. For example, given a function f(x) = x2 – 4,
In the interval [-2, 2], we derive the maximum and lowest values as follows:

Step 1: Differentiate the Function

Given,

- $f(x) = x^2 - 4$
- f'(x) = 2x

Step 2: Find out Critical Points

The critical points of the function can be found by setting f'(x) = 0. For the function in question,
2x = 0 ⇒ x = 0
Consequently, a critical point for this function is x = 0. It is now necessary to determine if it is a minimum or a maximum.

Step 3: Test for the Second Derivative

To determine if the provided critical point is a minima or a maxima, we apply the second derivative test previously discussed. For the example above, f''(x) = 2. You'll see that f''(x) > 0.

Step 4: Value at Critical Points

For the function f(0) = (0)2 – 4 = -4 the minimum value is -4

Thus, the minimum value of f(x) is at x = 0 is -4.

Applications of Relative Maxima and Minima

Applications for Relative Maxima and Minima are numerous. It serves a number of functions, including

With this idea, a stock's maximum and lowest values at specific times can be calculated by equating it to a function that depicts the stock's trajectory.

• Electronic circuits use this idea to control the system's voltage and current.

• Astrophysics also uses relative maxima and minima to determine the maximum and minimum trajectories of an object, etc.

Absolute Minima and Maxima

The function defined in a given interval has an absolute maximum and minimum value. Generally speaking, a function can have high or low values as we progress through it. The maxima and minima are the highest and lowest values of a function, respectively, in any given interval. The Absolute Maxima and Absolute Minima of the function are these maxima and minima if they are specified on the entire function.

This article will teach us everything there is to know about Absolute Maxima and Minima, including how to calculate them, their examples, and much more.

Definition 1.1

Absolute minimum and maximum values

If all x in the domain of f have f(c)≥f(x), then fat c has an absolute maximum. In the domain, the value f(c) is referred to as the maximum value off. Similar to this, the number f(c) is referred to as the minimum value of f on the domain, and an absolute minimum of fat c is defined as f(c)≤f(x) for every x in the domain of f. Extreme values of f are its maximum and minimum values.

The largest and lowest values of a function across a specified range are its absolute maxima and minima. Absolute Maxima and Minima are the highest and lowest values that the function can reach throughout its whole domain. They are sometimes referred to as the global maxima and minima of the function. Assume that f(x) = sinx is a function defined on interval R such that -1 < sin x ≤ 1.

As a result, f(x) has a maximum value of 1 and a minimum value of -1. As a result, 1 and -1 are the absolute maxima and minima of f(x) = sin x defined over R.

Critical Points and Extreme Value Theorem

Assuming we have a function f(x), the critical points are those at which the function's derivative approaches zero. These locations may be minima or maxima. A crucial point is where the second derivative test determines the minima or maxima. More than one minima or maxima are possible since the function's derivative may be zero at more than one place. There are several key moments in the function depicted in the image below.

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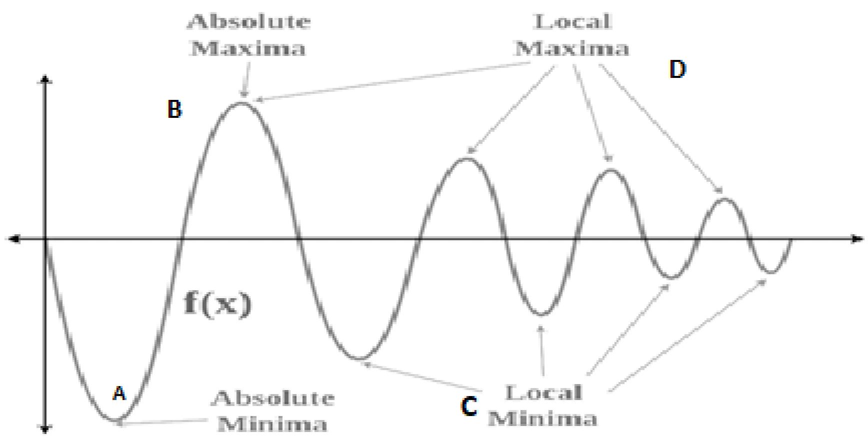

Figure 1.3 Absolute and Local Points of the Maxima & Minima

See Figure 1.3, where points B and D represent maxima and points A and C represent minima. Local minima and maxima are terms used to describe B and C, respectively. This indicates that while these positions may be maximum and minimum locally, they may not necessarily be so globally. Global minima and global maxima are points A and D.

Suppose we have a twice differentiable function f(x). The formula for its critical points is f'(x) = 0. We may determine if the determined critical point is a minima or a maximum by using the Second Derivative Test.

• A maximum is at point x if f''(x) > 0.

• The point x is the minimum if f''(x) < 0.

Although this test now identifies the minimum and maximum points, it is still unable to provide information regarding the global maxima and global minima. We are saved by the extreme value theorem.

Extreme Value Theorem

Under specific circumstances, the extreme value theorem guarantees a function's minima as well as its maxima. This theorem merely indicates that an extreme value will exist; it does not indicate where the extreme points will be. According to the theorem, function f(x) has at least one maximum and lowest value on [a, b] if it is continuous on the closed interval [a, b].

Absolute Minima and Maxima in Closed Interval

Now we need to take a few simple procedures to determine the extreme points in any interval. For example, suppose we have a region D and a function f(x). The function's extreme value within the interval is obtained.

First, determine the critical points of the function in interval D such that f'(x) = 0.
Step 2: Calculate the function's value at the interval D's extremes.
Step 3: The greatest and smallest values discovered in the previous two steps represent the function's absolute maximum and minimum.

.

Absolute Minima and Maxima in Entire Domain

The greatest and lowest values of the function, wherever it is specified, are the absolute minimum and maximum values of the function throughout the domain. A function may have a maximum value, a minimum value, or neither of them. A straight line, for instance, has no minimum or maximum value because it stretches to infinity in both directions.
To determine the absolute maxima and minima for the full domain, we must take some actions that are identical to those in the preceding example. minimal level of functionality

Step 1: Whenever the function is defined, identify its crucial points.
Step 2: Determine the function's value at these extreme locations.

Step 3: Find the function's value as x trends toward and away from infinity. Additionally, look for any discontinuities.

Step 4: The absolute maximum and absolute minimum for the function throughout its whole domain are given by the maximum and minimum of all these values.

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Local Maxima and Minima

The function's maximum and minimum values in relation to other locations inside a certain function interval are known as the local maxima and local minima. Generally, they are computed using the same formula as absolute maxima and minima. Any function's local maxima and minima can resemble or not resemble the function's absolute maxima and minima. Assuming that the function $f(x) = \cos x$ is defined on $[-\pi, \pi]$, its highest value is 1 and its smallest value is -1; these represent the function's local maxima and minima. The function's maximum and minimum values, or absolute maxima and minima, for the function f(x) defined on R are now 1 and -1. In this case,

- If there is an open interval (a,b) containing c such that f(c).≥.f(x) for each $x \in (a,b)$, then the f function has a local maximum (or relative maximum) at c.
- Similarly, if there is an open interval (a,b) containing c such that $f(c) \leq f(x)$ for all x! (a,b), then f has a local minimum at c.

Note

Global maximum and global minimum of a function f in (a,b) are another name for the absolute maximum and absolute minimum values of the function on an interval (a,b).

Standards for local minimums and maximums

Assume that f'' (c) exists and that f is a differentiable function on the open interval (a,b) that contains c.

(i) Local minimum of f at c if $f'(c) = 0$ and $f''(c) > 0$.

(ii) Local maximum of f at c if $f'(c) = 0$ and $f''(c) < 0$.

Note

In the field of economics, the maximum or lowest for cost or income is determined by the point at which dy/dx = 0, where y = f(x) represents the cost or revenue function.

Example 9:

Find Extremum values of $f(x) = 2x^3+3x^2-12x$.

Solution:

Given

$f(x) = 2x^3+3x^2-12x$ … (1)

$f'(x) = 6x^2+6x-12$

$f''(x) = 12x + 6$

$f'(x) = 0 \Longrightarrow 6x^2+6x-12 = 0$

$\Longrightarrow 6(x^2+x-2) = 0$

$\Longrightarrow 6(x+2)(x-1) = 0$

$\Longrightarrow x = -2$ and $x = 1$

When

$x = -2$

$f''(-2) = 12(-2) + 6$

$= -18 < 0$

$f(x)$ attains local maximum at $x = -2$ and local maximum value is obtained from (1) by substituting the value $x = -2$

$f(-2) = 2(-2)^3+3(-2)^2-12(-2)$

$= -16+12+24$

$= 20.$

When

x = 1

$f''(1) = 12(1) + 6$

$= 18.$

$f(x)$ attains local minimum at $x = 1$ and the local minimum value is obtained by substituting $x = 1$ in (1).

$f(1) = 2(1) + 3(1) - 12(1)$

$= -7$

Extremum values are – 7 and 20.

Example 10:

Find the absolute (global) maximum and absolute minimum of the function $f(x)=3x^5-25x^3+60x+1$ in the interval [–2,2]

Solution :

$f(x) = 3x^5-25x^3+60x+1$ … (1)

$f'(x) = 15x^4-75x^2+60$

$= 15(x^4-5x^2+4)$

$f'(x) = 0 \Longrightarrow 15(x^4-5x^2+4)= 0$

$\Longrightarrow (x^2-4)(x^2-1)= 0$

$x = \pm 2$ (or) $x = \pm 1$

of these four points $-2, \pm 1 \in [-2, 1]$ and $2 \notin [-2,1]$

From (1)

$f(-2) = 3(-2)^5 - 25(-2)^3 + 60(-2) + 1$

$= -15$

When $x= 1$

$f(1) = 3(1)^5 - 25(1)^3 + 60(1) + 1$

$= 39$

When $x = -1$

$f(-1) = 3(-1)^5 - 25(-1)^3 + 60(-1) + 1$

$= -37$.

Absolute maximum is 39

Absolute minimum is -37

1.9 Euler's theorem on homogeneous functions

Examine the following expression: $(x, y) = a0xn + a1xn-1y + a2xn-2y2 + \cdots + anyn$ First of all

Every term in the phrase above has a degree of "n." We refer to this kind of expression as a homogeneous function of degree 'n'. If f(x,y) = fn(x,y) for every x,y, and y, then f(x,y) is a homogeneous function of degree 'n' in y and y.

Theorem

Let f(x,y) be a homogeneous function of order n so that

- Euler's theorem states that if $f(x_1, x_2, \ldots., x_n)$ is a homogeneous function of degree k, then: $\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = kf(x_1, x_2, \ldots., x_n)$
- In other words, the sum of the products of each variable and its partial derivative with respect to that variable is equal to k times the original function.

Total derivatives

Definition 1.2

The total derivative of a function of multiple variables represents how the function changes with respect to changes in all of its variables, not just one at a time.

Suppose we have a function $f(x_1, x_2, \ldots., x_n)$ which depends on n variables. The total derivative of f with respect to a variable x_i (where i ranges from 1 to n) is denoted as $\frac{df}{dx_i}$ and is defined as:

$$\frac{df}{dx_i} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dx_i} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dx_i} + \cdots \ldots.. + \frac{\partial f}{\partial x_n}\frac{dx_n}{dx_i}$$

Here, $\frac{df}{dx_i}$ represents the total derivative of f with respect to x_i, and $\frac{dx_j}{dx_i}$ represents the rate of change of variable x_j with respect to x_i.

1.10 Lagrange's method of undetermined multipliers

The Lagrange's function can be formed as follows to find the minimum or maximum value of a function f(x) according to the equality constraint g(x) = 0.

f(x) – λg(x) = L(x, λ)

In this case, λ = Lagrange multiplier and L = Lagrange function of the variable x.

Important Notes on Partial Derivatives:

When determining the partial derivative of a variable, it is important to treat all other variables as constants.

• With partial derivatives, the sequence in which derivatives are taken is irrelevant. That is, −2f/∂x−y = −2f/∂y−x.

• Partial differentiation is also subject to the derivatives rules.

• Applying derivative formulae simplifies the process of determining the partial derivatives as opposed to using the limit definition.

1.11 Summary

A partial derivative of a function of multiple variables measures how the function changes with respect to changes in one variable while keeping all other variables constant. Partial derivatives are denoted using the ∂ symbol, which represents a partial differentiation. In geometric terms, the partial derivative represents the slope of the tangent line to the surface defined by the function in the direction of the specified variable. Understanding partial derivatives is fundamental for analyzing functions of multiple variables and solving problems in various fields involving interdependent quantities. They provide a powerful tool for studying the behavior of functions in complex systems and optimizing their performance.

1.12 Keywords

- Partial Derivative
- Higher order partial derivative
- Rules of Partial derivation
- Maxima and Minima
- Lagrange's method

1.13 Self – Assessment Questions

1. Find $\frac{\partial f}{\partial x}$ for $f(x,y) = 3x^2y + 2xy^3$.
2. Compute $\frac{\partial g}{\partial y}$ for $g(x,y) = e^{xy} + \ln(y)$.
3. Find $\frac{\partial h}{\partial x}$ for $h(x,y) = \sin(xy) + x\cos(y)$.

4. Compute $\frac{\partial f}{\partial y}$ for $f(x,y) = \frac{x^2}{y} + \sqrt{y}$.
5. Find $\frac{\partial g}{\partial x}$ for $g(x,y) = \frac{1}{x} + y^2$.
6. Compute $\frac{\partial h}{\partial y}$ for $h(x,y) = e^{xy} \cdot \cos(x)$.
7. Find $\frac{\partial f}{\partial x}$ for $f(x,y) = \ln(x) + e^{2y}$.
8. Compute $\frac{\partial g}{\partial y}$ for $g(x,y) = \frac{x}{y} + \sin(xy)$.
9. Find $\frac{\partial h}{\partial x}$ for $h(x,y) = \frac{\sqrt{x}}{y} + \frac{1}{x^2}$.
10. Compute $\frac{\partial f}{\partial y}$ for $f(x,y) = x^3y^2 + \frac{1}{y}$.

1.14 Case Study

1. A metal sheet measuring twelve meters by eight meters is to be used to build a rectangular box with an open top. Determine the box's dimensions that will require the least quantity of metal.
2. How may partial differentiation be used to address the optimization issue mentioned in section (a)?

1.15 References

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2. Strang, G. (2005). Introduction to Linear Algebra (4th ed.). Wellesley-Cambridge Press.

UNIT - 2

Separation of Variables

Learning Objectives:

- To understand Differential equations
- To understand Equations of first order and first degree
- To understand variable separable differentiation

Structure

2.1 Method of Separation of Variables

2.2 Introduction and Overview

2.3 Application to One-Dimensional Wave Equation and Diffusion Equation

2.4 Geometrical Interpretation

2.5 Summary

2.6 Keywords

2.8 Self Assessment questions

2.9 Case Study

2.9 References

2.1 Method of Separation of Variables

In some situations, a strong technique—the method of separation of variables—can be used to discover solutions to linear partial differential equations of order two with specified initial and boundary conditions. This approach starts with the assumption that the partial differential equation is initially trial solved as a product of several functions, each of which is a function of a single independent variable. This approach is sometimes referred to as the "product method" for this reason. The following specific example serves as the best way to demonstrate the procedure of separation of variables:

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=G$$

let A, B, C,...., G be the generic linear partial differential equation, where x and y are the functions involved. It should be observed that (i) A partial differential equation of order two is considered non-linear if it cannot be expressed in the above form as provided by the equation.

(ii) The partial differential equation shown above is considered homogeneous if G=0; otherwise, it is not. Therefore, we observe that G does not equal 0 for non-homogeneous partial differential equations.

Example 1: Solve Using method of separation of variables.

$$\frac{\partial^2 u}{\partial x^2}-2\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0$$

Solution:

$$\frac{\partial^2 u}{\partial x^2}-2\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0$$

$$u(x,y)=X(x)Y(y) \quad I\ \frac{\partial u}{\partial x}=X'Y\ ,\frac{\partial u}{\partial y}=XY' \text{ and } \frac{\partial^2 u}{\partial x^2}=X$$

X -2-λX= or $\left(D^2 - 2D - \lambda\right) X = 0$

$Y' - \lambda Y = 0$ or $\frac{dY}{dy} = \lambda Y$

$m^2 - 2m - \lambda = 0$, which gives $m = 1 \pm \sqrt{(1+\lambda)}$

Thus, we have $X(x) = Ae^{\left[1+\sqrt{1+\lambda}\right]x} + Be^{\left[1-\sqrt{1+\lambda}\right]x}$

$Y(y) = Ce^{\lambda y}$

$$u(x,y) = \left[ACe^{\left(1+\sqrt{(1+\lambda)}\right)x} + BC\,e^{\left(1-\sqrt{(1+\lambda)}\right)x}\right]e^{\lambda y}$$

$$u(x,y) = \left[A_1 e^{\left(1+\sqrt{(1+\lambda)}\right)x} + A_2 e^{\left(1-\sqrt{(1+\lambda)}\right)x}\right]e^{\lambda y}$$

2.2 Introduction and Overview

The method of separation of variables is a powerful technique commonly used to solve partial differential equations (PDEs), particularly those that are linear and homogeneous, with constant coefficients. This method exploits the assumption that the solution to the PDE can be expressed as a product of functions, each depending on only one of the independent variables. Let's delve into an introduction and overview of this method. Partial differential equations arise in various scientific and engineering disciplines to describe physical phenomena involving multiple independent variables. Solving these equations allows us to understand and predict the behavior of systems governed by them. The method of separation of variables provides a systematic approach to finding solutions to certain types of partial differential equations.

The method of separation of variables involves the following general steps:

1. **Formulation of the PDE**: Begin with a given partial differential equation (PDE) that needs to be solved. This equation typically describes the relationship between the dependent variable and its independent variables, along with any given boundary or initial conditions.

2. **Assumption of Separable Solution**: Assume that the solution to the PDE can be expressed as a product of functions, each depending on only one of the independent variables.
3. **Substitution and Separation**: Substitute the assumed separable solution into the original PDE and then separate the variables by moving all terms involving each variable to one side of the equation. This results in a set of ordinary differential equations (ODEs), each involving only one independent variable and its corresponding function.
4. **Solving the ODEs**: Solve each resulting ODE individually to find the solutions for the functions. This step usually involves finding eigen values and eigen functions or applying other techniques appropriate to the specific ODE.
5. **Combining Solutions**: After obtaining the solutions for each function, combine them using the assumed separable form to form the general solution to the original PDE.
6. **Applying Boundary or Initial Conditions**: If boundary or initial conditions are provided with the problem

2.3 Application to one-dimensional wave equation and diffusion equation

Consider a P.D.E

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \;\; 0 < t < \infty$$

with the boundary conditions (**BCs**):

$$\begin{cases} u(0,t) = 0 \\ u(L,t) = 0 \end{cases}$$

and the initial condition (**IC**):

$$u(x,0) = \varphi(x), \quad 0 \leqslant x \leqslant L$$

2.4 Geometrical Interpretation

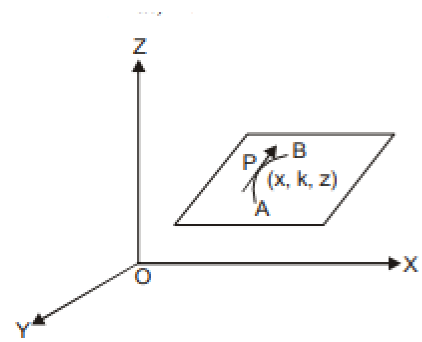

Figure 2.1 : Geometrical Interpretation of Z

Given a surface S, let z = f(x, y).

Let y = k be a plane that cuts the surface Z = f(x, y) along the curve APB, passing through P (x, k, z). This plane is parallel to the XZ - plane.

The equations for the plane curve in section APB are z = f(x, y) y = k.

The slope of the tangent to this curve is given by

$$\frac{\partial z}{\partial x}.$$

2.5 Summary

By dissecting the issue into more manageable ordinary differential equations (ODEs), separation of variables is a potent strategy for solving partial differential equations (PDEs). The essential concept is to presume that the PDE's solution may be written as the product of functions, each of which depends only on one variable.

2.6 Keywords

- Method of Separation
- Product Solutions
- Fourier Series
- Laplace's Equation
- Heat Equation
- Wave Equation
- Diffusion Equation

2.7 Self-Assessment Questions

1. Define Separation of Variables and explain its significance in solving partial differential equations.
2. Describe the general procedure for applying Separation of Variables to solve a partial differential equation.
3. What are the key assumptions made when applying Separation of Variables?
4. Provide an example of a partial differential equation problem where Separation of Variables can be effectively applied.
5. How do you determine the separated solutions for each variable in the Separation of Variables method?

2.8 Case Study

A non-mathematical context: designing a multi-stage distillation column for separating components in a chemical process.

Problem Statement: Imagine a chemical engineering company tasked with designing a distillation column to separate a mixture of ethanol and water into its pure components. The goal is to achieve high purity ethanol as the top product and high purity water as the bottom product.

2.9 References

1. William. E.(2017). Elementary Differential Equations and Boundary Value Problems. John Wiley & Sons.
2. Erwin Kreyszig (2020). Advanced Engineering Mathematics. John Wiley & Sons

UNIT- 3
First Order Partial Differential Equations (PDEs)

Learning Objectives:

- Define partial differential equations (PDEs) of second order and distinguish them from other types of PDEs, such as first-order PDEs.
- Identify and derive the canonical forms of second-order PDEs, such as the Laplace, heat, and wave equations, in various coordinate systems.
- Discuss numerical techniques, such as finite difference, finite element, and spectral methods, for approximating solutions to PDEs.

Structure

3.1 Partial differential Equations:-

An equation with z as the dependent variable and x, y, and z as the independent variables, such that z = f(x,y), is called a partial differential equation. We also use the notations

$$\frac{\partial z}{\partial x} = p,\ \frac{\partial z}{\partial y} = q,\ \frac{\partial^2 z}{\partial x^2} = r,\ \frac{\partial^2 z}{\partial x \partial y} = s,\ \frac{\partial^2 z}{\partial y^2} = t.$$

3.2 Order of a Partial differential Equations

A partial differential equation's order is determined by the highest partial derivative that appears in the equation, and its degree is determined by that derivative's degree.

For example, (1) x + y

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$$

With z as the dependent variable and x and y as the independent variables, this equation has order one and degree one.

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 z}{\partial^2 y} + \frac{\partial^2 v}{\partial z^2} = 0$$

This is an order two, degree equation where v is dependent and x, y, and z are independent. The study of wave equations, heat equations, electromagnetic, radar, ratios, television, and other subjects will all heavily rely on partial differential equations.

Formation of Partial Differential Equation(PDE)

PDE can be obtained by:

(i) Doing away with random constants
(ii) Removing any functions that include two or more variables at random.

Elimination of Arbitrary Constants

Let f(x, y, z,a,b) = 0 ...(1)

consist of an equation with the two arbitrary constants "a" and "b." Partially differentiating this equation in relation to x and y yields

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\left(\frac{\partial z}{\partial x}\right) = 0 \qquad \ldots(2)$$

3.3 Lagrange's Method and Standard Forms

Partial differential equations of Lagrange's kind are those that have the formula Pp + Qq = R.

Steps for solving Pp+Qq=R by Lagrange's method

STEP 1: Insert the first-order linear partial differential equation that has been provided in the standard from

Pp+Qg=R ………..(1)

STEP 2: Note down the following Lagrange's auxilliary equation for (1):

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \ldots\ldots\ldots\ldots(2)$$

STEP 3: Apply the established techniques to solve (2). As two independent solutions of (2), let u(x, y, z) = c1 and v(x, y, z) = c_2.

STEP 4: The general solution, also known as the integral, of (1) is then expressed in one of the three equivalent ways shown below: φ(u,v)=0, where \ is an arbitrary function and u= φ(v) or v= φ(u).

Example 1: Solve a(p+q) = z

Solution: Given $ap + aq = z$

The Lagrange's Auxiliary equation

$$\frac{dx}{a} = \frac{dy}{a} = \frac{dz}{1}$$

$$dx - dy = 0$$

Integrating

$x - y = c_1$

Taking the last two members

$$dy - a\,dz = 0$$

Integrating

$$y - az = c2$$

the required solution is given by

ϕ(x- y, y – az) = 0, ϕ being an arbitrary function.

Example 2: Solve

(mz - ny)p + (nx - lz)q = ly – mx

Sol. The Lagrange's auxiliary equation of the given equation are

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}$$

Changing x, y, z as multipliers, each fraction

$$= \frac{xdx + ydy + zdz}{x(mz - ny) + y(nx - lz) + z(ly - mx)}$$

$$= \frac{xdx + ydy + zdz}{0}$$

Therefore $xdx + ydy + zdz = 0$

or $2xdx + 2ydy + 2zdz = 0$

Integrating, $x^2 + y^2 + z^2 = c_1$.

$$= \frac{ldx + mdy + ndz}{l(mz - ny) + m(nx - lz) + n(ly - mx)}$$

$$= \frac{ldx + mdy + n\,dz}{0}$$

Therefore $\quad ldx + mdy + n\,dz = 0$

so that $\quad lx + my + nz = c_2$

the required general solution is given by

$$\phi(x^2 + y^2 + z^2,\ ldx + mdy + n\,dz) = 0$$

Example 3: Solve

$$\left(\frac{y^2z}{x}\right)p + zxq = y^2$$

Solution: The Lagrange's auxiliary equations are

$$\frac{dx}{\frac{y^2z}{x}} = \frac{dy}{zx} = \frac{dz}{y^2}$$

or $$\frac{xdx}{y^2z} = \frac{dy}{zx} = \frac{dz}{y^2}$$

From 1st and 2nd fractions we get

$$\frac{xdx}{y^2z} = \frac{dy}{zx}$$

or $$x^2dx = y^2dy$$

integrating

$$\frac{x^3}{3} = \frac{y^3}{3} + \frac{c_1}{3}$$

$$c_1 = x^3 - y^3$$

From 1st and 3rd fractions we get

$$\frac{xdx}{y^2z} = \frac{dz}{y^2}$$

or $\quad xdx = zdz$

integrating

or $\quad \frac{x^2}{2} = \frac{z^2}{2} + \frac{c_2}{2}$

or $\quad c_2 = x^2 - z^2$

The general solution is given by

$$f(c_1, c_2) = 0$$

$$f(x^3 - y^3, x^2 - z^2) = 0$$

3.4 Charpit's Method

In the event that the provided equation cannot be reduced to one of the four first-order non-linear partial differential equation types mentioned above, we solve all first-order partial differential equations using a method developed by Charpit. This method is known as Charpit's method.

$$F(x, y, z, p, q) = 0$$

Since z depend on x and y, we have,

$$dZ = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = Pdx + Qdy$$

Now, if we can find another relation between x, y, z, p, q such that,

$$f(x, y, z, p, q) = 0$$

Example 4 :

Solve $(p^2 + q^2)y = qz$.

Solution: Let $F(x, y, z, p, q) = (p^2 + q^2)y - qz = 0$ ...(1)

The subsidiary equations are:

$$\frac{dx}{-2py} = \frac{dy}{z-2qy} = \frac{dz}{-qz} = \frac{dp}{-pq} = \frac{dq}{p^2}$$

The last two fractions yield $pdp + qdq = 0$

which on integration gives

$p^2 + q^2 = c^2$ ...(2)

In order to solve equations (1) and (2), put $p^2 + q^2 = c^2$ in equation (1) so that $q = \frac{c^2 y}{z}$

Now, substituting this value of q in equation (2), we get

$p = c\sqrt{\frac{z^2 - c^2y^2}{z}}$

Hence, $dz = pdx + qdy = \frac{c}{z}\sqrt{(z^2 - c^2y^2)dx} + \frac{c^2y}{z}dy$

$\Rightarrow zdz - c^2ydy = c\sqrt{z^2 - c^2y^2}dx$

$\Rightarrow \frac{(1/2)d(z^2 - c^2y^2)}{\sqrt{z^2 - c^2y^2}} = cdx$

Integrating, we get the required solution as $z^2 = (a + cx)^2 + c^2y^2$

Example 5:

Solve $2zx - px^2 - 2qxy + pq = 0$.

Solution: Let $F = 2zx - px^2 - 2qxy + pq = 0$

The Charpit's auxillary equations are:

$$\frac{dx}{\frac{-\partial F}{\partial p}} = \frac{dy}{\frac{-\partial F}{\partial q}} = \frac{dz}{-p\frac{\partial F}{\partial p} - q\frac{\partial F}{\partial q}} = \frac{dp}{\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z}} = \frac{dq}{\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z}} = \frac{df}{0}$$

Here, $\frac{dp}{2z - 2ay} = \frac{dq}{0} = \frac{dz}{px^2 - pq + 2xyq - pq} = \frac{dx}{x^2 - q} = \frac{dy}{2xy - p} = \frac{dF}{0}$

$\therefore dq = 0 \Rightarrow q = a$

Substituting $q = a$ in the given equation, we have

$2zx - px^2 - 2axy + pa = 0$

$p(x^2 - a) = 2x(z - ay)$

$\Rightarrow p = \frac{2x(z - ay)}{x^2 - a}$

Substituting these values of p, q in dz, we have

$$dz = \frac{2x(z-ay)}{x^2-a}dx + ady$$

$$\frac{dz-ady}{z-ay} = \frac{2xdx}{x^2-a}$$

Integrating, we get

$$\log(z - ay) = \log(x^2 - a) + \log b$$

$$=> z - ay = b(x^2 - a)$$

$=> z = ay + b(x^2 - a)$ which is the complete integral of the given equation.

3.5 Summary

Understanding and solving second-order PDEs is crucial for modeling and analyzing complex physical systems, predicting their behavior, and designing optimal engineering solutions. These equations play a fundamental role in diverse areas of science and technology, contributing to advancements in fields ranging from aerospace engineering to medical imaging.

3.6 Keywords

- Partial differential Equations
- Order of a Partial differential Equations
- Lagrange's Method
- Charpit's Method

3.7 Self-Assessment Questions

1. How do boundary value problems (BVPs) and initial value problems (IVPs) arise in the context of second-order PDEs, and what techniques are used to solve them?
2. What role do boundary conditions and initial conditions play in determining solutions to second-order PDEs?
3. Can you explain the concept of characteristic curves or surfaces in the context of hyperbolic and parabolic second-order PDEs?
4. How do numerical methods, such as finite difference, finite element, and spectral methods, contribute to the solution of second-order PDEs?

3.8 Case Study

1. Which of the following represents a linear partial differential equation?

A) $u_{xx} + u_{yy} = u$

B) $u_{xx} - u_{yy} = \sin(xy)$

C) $u_x + u_y = u^2$

D) $u_x u_y = u$

2. The heat equation, describing the flow of heat in a given region over time, is an example of which type of partial differential equation?

A) Elliptic

B) Parabolic

C) Hyperbolic

D) Bilinear

3. Which of the following methods is commonly used to solve homogeneous linear partial differential equations with constant coefficients?

A) Method of characteristics

B) Fourier transform

C) Separation of variables

D) Laplace transform

4. Consider the partial differential equation $u_{tt} - c^2 u_{xx} = 0$, where c is a constant. What type of equation is this?

A) Parabolic

B) Hyperbolic

C) Elliptic

D) Transcendental

3.9 References

1. Evans, L. C. (2020). Partial Differential Equations (2nd ed.). American Mathematical Society.
2. Strauss, W. A. (2018). Partial Differential Equations: An Introduction. John Wiley & Sons.

UNIT - 4
Linear Differential Equations

Learning Objectives:

- To understand linear Differential equations
- To understand Equations of first order and first degree
- To understand Bernoulli equation

Structure:

4.1 Linear Differential Equations

4.2 Bernoulli equation

4.3 Summary

4.4 Keywords

4.5 Self-Assessment Questions

4.6 Case Study

4.7 References

4.1 Linear Differential Equations

A differential equation of the form

$$\boxed{\frac{dy}{dx}+Py=Q}$$

Working Rule

Step 1. Convert the given equation to the standard form of linear differential equation

i.e. $$\frac{dy}{dx}+Py=Q$$

Step 2. Find the integrating factor i.e. I.F. = $e^{\int P dx}$

Step 3. Then the solution is $y(I.F.)=\int Q\ (I.F.)dx+C$

Ex.1. Evaluate

$$(x+1)\frac{dy}{dx}-y=e^x(x+1)^2$$

Solution. $$\frac{dy}{dx}-\frac{y}{x+1}=e^x(x+1)$$

$$\text{I.F.} = e^{-\int\frac{dx}{x+1}}=e^{-\log(x+1)}=e^{\log(x+1)^{-1}}=\frac{1}{x+1}$$

$$y.\frac{1}{x+1}=\int e^x.(x+1).\frac{1}{x+1}dx=\int e^x dx$$

$$\frac{y}{x+1}=e^x+C$$ **Ans.**

Ex 2. Evaluate

$$(x^3-x)\frac{dy}{dx}-(3x^2-1)y=x^5-2x^3+x.$$

Solution. We have $(x^3-x)\frac{dy}{dx}-(3x^2-1)y=x^5-2x^3+x$

$$\Rightarrow \quad \frac{dy}{dx}-\frac{3x^2-1}{x^3-x}y=\frac{x^5-2x^3+x}{x^3-x} \quad \Rightarrow \quad \frac{dy}{dx}-\frac{3x^2-1}{x^3-x}y=x^2-1$$

$$\text{I.F.} = e^{\int -\frac{3x^2-1}{x^3-x}dx}=e^{-\log(x^3-x)}=e^{\log(x^3-x)^{-1}}=\frac{1}{x^3-x}$$

Its solution is

$$y(\text{I.F.})=\int Q(I.F.)\,dx+C \quad \Rightarrow \quad y\left(\frac{1}{x^3-x}\right)=\int\frac{x^2-1}{x^3-x}dx+C$$

$$\Rightarrow \quad \frac{y}{x^3-x}=\int\frac{x^2-1}{x(x^2-1)}dx+C \quad \Rightarrow \quad \frac{y}{x^3-x}=\int\frac{1}{x}dx+C$$

$$\Rightarrow \quad \frac{y}{x^3-x}=\log x+C \quad \Rightarrow \quad y=(x^3-x)\log x+(x^3-x)\,C$$ **Ans.**

Ex. 3. Evaluate

$$\sin x\frac{dy}{dx}+2y=\tan^3\left(\frac{x}{2}\right)$$

Solution. Given equation : $\sin x\frac{dy}{dx}+2y=\tan^3\frac{x}{2} \Rightarrow \frac{dy}{dx}+\frac{2}{\sin x}y=\frac{\tan^3\frac{x}{2}}{\sin x}$

This is linear form of $\frac{dy}{dx}+Py=Q$

$\therefore$ $$P=\frac{2}{\sin x} \text{ and } Q=\frac{\tan^3\frac{x}{2}}{\sin x}$$

$\therefore$ $$\text{I.F.}=e^{\int P dx}=e^{\int\frac{2}{\sin x}dx}=e^{2\int \operatorname{cosec} x\,dx}=e^{2\log\tan\frac{x}{2}}=\tan^2\frac{x}{2}$$

$\therefore$ Solution is $$y.(\text{I.F.})=\int \text{I.F.}(Q\,dx)+C$$

$$y\tan^2\frac{x}{2}=\int\tan^2\frac{x}{2}\cdot\frac{\tan^3\frac{x}{2}}{2\sin\frac{x}{2}\cdot\cos\frac{x}{2}}+C=\frac{1}{2}\int\frac{\tan^4\frac{x}{2}}{\cos^2\frac{x}{2}}dx+C$$

$$=\frac{1}{2}\int\tan^4\frac{x}{2}.\sec^2\frac{x}{2}dx+C \quad \text{... (1)}$$

Putting $\tan\frac{x}{2}=t$ so that $\frac{1}{2}\sec^2\frac{x}{2}dx=dt$ on R.H.S. (1), we get

$$y.\tan^2\frac{x}{2}=\frac{1}{2}\int t^4(2dt)+C \Rightarrow y\tan^2\frac{x}{2}=\frac{t^5}{5}+C$$

$$y\tan^2\frac{x}{2}=\frac{\tan^5\frac{x}{2}}{5}+C$$ **Ans.**

Exercise

Solve:

1. $\frac{dy}{dx}+\frac{1}{x}y=x^3-3$ **Ans.** $xy=\frac{x^5}{5}-\frac{3x^2}{2}+C$

2. $(2y-3x)\,dx+x\,dy=0$ **Ans.** $yx^2=x^3+C$

3. $\frac{dy}{dx}+y\cot x=\cos x$ **Ans.** $y\sin x=\frac{\sin^2 x}{2}+C$

4. $\frac{dy}{dx}+y\sec x=\tan x$ **Ans.** $y=\frac{C-x}{\sec x+\tan x}+1$

5. $\cos^2 x\frac{dy}{dx}+y=\tan x$ **Ans.** $y=\tan x-1+Ce^{-\tan x}$

6. $(x+a)\frac{dy}{dx}-3y=(x+a)^5$ **Ans.** $2y = (x+a)^5 + 2C(x+a)^3$

7. $x\cos x\frac{dy}{dx}+y(x\sin x+\cos x)=1$ **Ans.** $x\,y = \sin x + C\cos x$

8. $x\log x\frac{dy}{dx}+y=2\log x$ **Ans.** $y\log x = (\log x)^2 + C$

9. $x\frac{dy}{dx}+2y=x^2\log x$ **Ans.** $yx^2=\frac{x^4}{4}\log x-\frac{x^4}{16}+C$

10. $dr+(2r\cot\theta+\sin 2\theta)\,d\theta=0$ **Ans.** $r\sin^2\theta=\frac{-\sin^4\theta}{2}+C$

4.2 Bernoulli equation:-

The equation of the form

$$\frac{dy}{dx}+Py=Qy^n \quad \text{...(1)}$$

where **P** and **Q** are constants or functions of x can be reduced to the linear form on dividing by y^n and substituting $\frac{1}{y^{n-1}}=z$

On dividing bothsides of (1) by y^n, we get

$$\frac{1}{y^n}\frac{dy}{dx}+\frac{1}{y^{n-1}}P=Q \quad \text{...(2)}$$

Put $\frac{1}{y^{n-1}}=z$, so that $\frac{(1-n)}{y^n}\frac{dy}{dx}=\frac{dz}{dx} \Rightarrow \frac{1}{y^n}\frac{dy}{dx}=\frac{dz}{1-n}$

$\therefore$ (2) becomes $\frac{1}{1-n}\frac{dz}{dx}+Pz=Q$ or $\frac{dz}{dx}+P(1-n)z=Q(1-n)$

Ex 4. Evaluate

$x^2dy + y(x+y)\,dx = 0$

Solution. We have, $x^2\,dy + y\,(x+y)\,dx = 0$

$$\Rightarrow \frac{dy}{dx}+\frac{y}{x}=-\frac{y^2}{x^2} \Rightarrow \frac{1}{y^2}\frac{dy}{dx}+\frac{1}{xy}=-\frac{1}{x^2}$$

Put $-\frac{1}{y}=z$ so that $\frac{1}{y^2}\frac{dy}{dx}=\frac{dz}{dx}$

The given equation reduces to a linear differential equation in z.

$$\frac{dz}{dx}-\frac{z}{x}=-\frac{1}{x^2}$$

$$\textbf{I.F.}=e^{-\int\frac{1}{x}dx}=e^{-\log x}=e^{\log 1/x}=\frac{1}{x}.$$

Hence the solution is

$$z.\frac{1}{x}=\int-\frac{1}{x^2}.\frac{1}{x}\,dx+C \quad\Rightarrow\quad \frac{z}{x}=\int -x^{-3}\,dx+C$$

$$\Rightarrow\quad -\frac{1}{xy}=-\frac{x^{-2}}{-2}+C \quad\Rightarrow\quad \frac{1}{xy}=-\frac{1}{2x^2}-C \qquad \textbf{Ans.}$$

Ex 5. Evaluate

$$x\frac{dy}{dx}+y\log y=xy\ e^x$$

Solution. $x\frac{dy}{dx}+y\log y=xy\,e^x$

Dividing by xy, we get

$$\frac{1}{y}\frac{dy}{dx}+\frac{1}{x}\log y=e^x \qquad ...(1)$$

Put $\log y=z$, so that $\frac{1}{y}\frac{dy}{dx}=\frac{dz}{dx}$

Equation (1) becomes, $\frac{dz}{dx}+\frac{z}{x}=e^x$

$$\text{I.F.}=e^{\int\frac{1}{x}dx}=e^{\log x}=x$$

Solution is

$$zx=\int xe^x\,dx+C$$

$$zx=xe^x-e^x+C$$

$$\Rightarrow\quad x\log y=xe^x-e^x+C \qquad \textbf{Ans.}$$

Ex 6. Evaluate

$$\frac{dy}{dx}-\frac{\tan y}{1+x}=(1+x)e^x\sec y.$$

Solution. $\frac{dy}{dx}-\frac{\tan y}{1+x}=(1+x)e^x\sec y$

$$\Rightarrow\quad \cos y\frac{dy}{dx}-\frac{\sin y}{1+x}=(1+x)e^x \qquad ...(1)$$

Put $\sin y=z$, so that $\cos y\frac{dy}{dx}=\frac{dz}{dx}$

(1) becomes $$\frac{dz}{dx}-\frac{z}{1+x}=(1+x)e^x$$

$$I.F.=e^{-\int\frac{1}{1+x}dx}=e^{-\log(1+x)}=e^{\log\frac{1}{1+x}}=\frac{1}{1+x}$$

Solution is $$z.\frac{1}{1+x}=\int(1+x)e^x.\frac{1}{1+x}dx+C=\int e^x dx+C$$

$$\frac{\sin y}{1+x}=e^x+C$$ **Ans.**

Ex 7 . Evaluate

$$\tan y\frac{dy}{dx}+\tan x=\cos y\cos^2 x$$

Solution. $$\tan y\frac{dy}{dx}+\tan x=\cos y\cos^2 x$$

$$\sec y\tan y\frac{dy}{dx}+\sec y\,\tan x=\cos^2 x$$

Writing $z=\sec y$, so that $\frac{dz}{dx}=\sec y\tan y\frac{dy}{dx}$

The equation becomes $$\frac{dz}{dx}+z\tan x=\cos^2 x$$

$$\text{I.F.}=e^{\int\tan x\,dx}=e^{\log\sec x}=\sec x$$

$\therefore$ The solution of the equation is

$$z\sec x=\int\cos^2 x\sec x\,dx+C$$

$$\sec y\sec x=\int\cos x\,dx+C=\sin x+C$$

$$\sec y=(\sin x+C)\cos x$$ **Ans.**

Ex 8. Evaluate

$$x\left[\frac{dx}{dy}+y\right]=1-y$$

Solution

which is in linear form of $\frac{dy}{dx}+Py=Q.$

$\therefore$ $$P=\left(1+\frac{1}{x}\right),\qquad Q=\frac{1}{x}$$

$$\text{I.F.}=e^{\int P dx}=e^{\int\left(1+\frac{1}{x}\right)dx}=e^{x+\log x}=e^x.e^{\log x}=e^x.x=xe^x$$

$$y(\text{I.F.})=\int\text{I.F.}(Q\,dx)+C$$

$$y(x.e^x)=\int(x.e^x)\times\frac{1}{x}dx+C\Rightarrow y(x.e^x)=\int e^x dx+C$$

$$y\,(x\,.\,e^x)=e^x+C$$

$$y=\frac{1}{x}+\frac{C}{x}e^{-x}$$ **Ans.**

Ex 9. Evaluate

$y \log y \, dx + (x - \log y) \, dy = 0$

Solution. We have,

$$y \log y \, dx + (x - \log y) \, dy = 0$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{-x + \log y}{y \log y} \quad \Rightarrow \quad \frac{dx}{dy} = \frac{-x}{y \log y} + \frac{\log y}{y \log y}$$

$$\Rightarrow \quad \frac{dx}{dy} + \frac{x}{y \log y} = \frac{1}{y}$$

$$\text{I.F.} = e^{\int \frac{1}{y \log y} dy} = e^{\log(\log y)} = \log y$$

Its solution is $x.\log y = \int \frac{1}{y} (\log y) \, dy$

$$x.\log y = \frac{(\log y)^2}{2} + C$$ **Ans.**

Ex 10. Evaluate

$(1 + y^2) \, dx = (\tan^{-1} y - x) \, dy.$

Solution. $(1 + y^2) \, dx = (\tan^{-1} y - x) \, dy$

$$\frac{dx}{dy} = \frac{\tan^{-1} y - x}{1 + y^2} \quad \Rightarrow \quad \frac{dx}{dy} + \frac{x}{1 + y^2} = \frac{\tan^{-1} y}{1 + y^2}$$

This is a linear differential equation.

$$\text{I.F.} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

Its solution is $x.e^{\tan^{-1} y} = \int e^{\tan^{-1} y} \frac{\tan^{-1} y}{1 + y^2} dy + C$

Put $\tan^{-1} y = t$ on R.H.S., so that $\frac{1}{1 + y^2} dy = dt$

$$x.e^{\tan^{-1} y} = \int e^t .t \, dt + C = t.e^t - e^t + C = e^{\tan^{-1} y} \left(\tan^{-1} y - 1 \right) + C$$

$$x = \left(\tan^{-1} y - 1 \right) + Ce^{-\tan^{-1} y}$$ **Ans.**

Ex 11. Evaluate

$r \sin \theta - \frac{dr}{d\theta} \cos \theta = r^2$

Solution. The given equation can be written as $-\frac{dr}{d\theta}\cos\theta + r\sin\theta = r^2$ ... (1)

Dividing (1) by $r^2\cos\theta$, we get $-r^{-2}\frac{dr}{d\theta} + r^{-1}\tan\theta = \sec\theta$ ... (2)

Putting $r^{-1} = v$ so that $-r^{-2}\frac{dr}{d\theta} = \frac{dv}{d\theta}$ in (2), we get

$$\frac{dv}{d\theta} + v\tan\theta = \sec\theta$$

$$\text{I.F.} = e^{\int \tan\theta\, d\theta} = e^{\log\sec\theta} = \sec\theta$$

Solution is $v\sec\theta = \int \sec\theta,\, \sec\theta + C \Rightarrow v\sec\theta = \int \sec^2\theta\, d\theta + C$

$$\frac{\sec\theta}{r} = \tan\theta + C \Rightarrow r^{-1} = (\sin\theta + C\cos\theta)$$

$\therefore$ $r = \frac{1}{\sin\theta + C\cos\theta}$ **Ans.**

4.3 Summary

A basic type of differential equations in mathematics, linear differential equations are used to simulate a variety of biological, physical, and economic systems. This is a brief overview that covers the main ideas, categories, approaches, and illustrations.

4.4 Keywords

- Linear Differential Equation
- Order
- First-order
- Second-order
- Homogeneous Equation

4.5 Self Assessment questions

1. $\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y} = 2xe^{-x}$ **Ans.** $e^x + x^2y + Cy = 0$

2. $3\frac{dy}{dx} + 3\frac{y}{x} = 2x^4y^4$ **Ans.** $\frac{1}{y^3} = x^5 + Cx^3$

3. $\frac{dy}{dx} = y\tan x - y^2\sec x$ **Ans.** $\sec x = (\tan x + C)\, y$

4. $\frac{dy}{dx} = 2y\tan x + y^2\tan^2 x$, if $y = 1$ at $x = 0$ **Ans.** $\frac{1}{y}\sec^2 x = -\frac{\tan^3 x}{3} + 1$

5. $\frac{dy}{dx} + \tan x \tan y = \cos x \sec y$ **Ans.** $\sin y \sec x = x + C$

6. $dy + y\tan x\,.\,dx = y^2 \sec x\,.\,dx$ **Ans.** $y(x + C) + \cos x = 0$

7. $(x^2y^2 + xy)\,y\,dx + (x^2y^2 - 1)\,x\,dy = 0$ **Ans.** $x\,y = \log C\,y$

8. $(x^2 + y^2 + x)\,dx + xy\,dy = 0$ **Ans.** $x^2y^2 = -\frac{x^4}{2} - \frac{2x^3}{3} + C$

9. $\frac{dy}{dx} + y = 3e^x y^3$ **Ans.** $\frac{1}{y^2} = 6e^x + Ce^{2x}$

10. $(x - y^2)\,dx + 2\,x\,y\,dy = 0$ **Ans.** $\frac{y^2}{x} + \log x = C$

4.6 Case Study

Rhythmic Mass, spring, and Damper Mechanism

Imagine you have a mass (m), a spring (k), and a damper (c) with a damping coefficient. This is a mass-spring-damper system. An external force (F(t)) causes the mass to shift from its equilibrium position by a distance (x(t)).

1. Determine the differential equation controlling the mass's motion by analyzing its motion.
2. Determine the specific solution when (F(t)), using a variety of techniques such variable parameters or unknown coefficients.

4.7 References

1. Kristensson, G. (2020). Second Order Differential Equations: Special Functions and Their Classification. Germany: Springer New York.
2. Keskin, A. Ü. (2018). Ordinary Differential Equations for Engineers: Problems with MATLAB Solutions. Germany: Springer International Publishing.

UNIT - 5
Homogeneous Linear System with Constant Coefficients

Learning Objectives:

- To understand Linear Homogeneous Differential equations
- To understand method of complementary function
- To understand rule of particular integral

Structure

5.1 Linear Homogeneous Differential Equations of second order with Constant Coefficients

The General form of D.E of second Order is given by

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

where Pand Q are constants and R is a function of x and D is diffential operator.

$$Dy = \frac{dy}{dx}, \quad D^2y = \frac{d^2y}{dx^2}$$

$\frac{1}{D}$ stands for the operation of integration.

$\frac{1}{D^2}$ stands for the operation of integration twice.

$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ can be written in the operator form.

$D^2y + P\,Dy + Q\,y = R \qquad \Rightarrow \qquad (D^2 + PD + Q)\,y = R$

Complete solution = Complementary function + Particular Integral

Let us consider a L.D.E of first order

$$\frac{dy}{dx} + Py = Q \qquad ...(1)$$

Its solution is $\quad ye^{\int P dx} = \int (Q\,e^{\int P dx})\,dx + C$

$\Rightarrow \quad y = Ce^{-\int P dx} + e^{-\int P dx}\int (Qe^{\int P dx})\,dx$

$\Rightarrow \quad y = cu + v\text{(say)} \qquad ...(2)$

where $u = e^{-\int P dx}$ and $v = e^{-\int P dx}\int Q\,e^{\int P dx}\,dx$

(*i*) Now differentiating $u = e^{-\int P dx}$ w.r.t. x, we get $\frac{du}{dx} = -Pe^{-\int P dx} = -Pu$

$\Rightarrow \quad \frac{du}{dx} + Pu = 0 \quad \Rightarrow \quad \frac{d(cu)}{dx} + P(cu) = 0$

which shows that $y = c.u$ is the solution of $\frac{dy}{dx} + Py = 0$

(*ii*) Differentiating $v = e^{-\int P dx}\int (Qe^{\int P dx}\,dx$ with respect to x, we get

$\frac{dv}{dx} = -Pe^{\int P dx}\int (Qe^{\int P dx})\,dx + e^{-\int P dx}Qe^{\int P dx} \quad \Rightarrow \quad \frac{dv}{dx} = -Pv + Q$

$\Rightarrow \quad \frac{dv}{dx} + Pv = Q$ which shows that $y = v$ is the solution of $\boxed{\frac{dy}{dx} + Py = Q}$

$\Rightarrow$ $$\boxed{y = C.F. + P.I.}$$

Rules for complementery function

(1) In finding the complementary function, R.H.S. of the given equation is replaced by zero.

(2) Let $y = C_1 e^{mx}$ be the C.F. of

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \qquad ...(1)$$

Putting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1) then $C_1 e^{mx}(m^2 + Pm + Q) = 0$

$\Rightarrow$ $m^2 + Pm + Q = 0$. It is called **Auxiliary equation**.

(3) Solve the auxiliary equation :

Case I : Roots, Real and Different. If m_1 and m_2 are the roots, then the C.F. is

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

Case II : Roots, Real and Equal. If both the roots are m_1, m_1 then the C.F. is

$$y = (C_1 + C_2 x)\, e^{m_1 x}$$

Equation (1) can be written as

$$(D - m_1)(D - m_1)y = 0 \qquad ... (2)$$

Replacing $(D - m_1)y = v$ in (2), we get

$$(D - m_1)v = 0 \qquad ... (3)$$

$$\frac{dv}{dx} - m_1 v = 0 \quad \Rightarrow \quad \frac{dv}{v} = m_1 dx \quad \Rightarrow \quad \log v = m_1 x + \log c_2 \quad \Rightarrow \quad v = c_2 e^{m_1 x}$$

$$v = c_2 e^{m_1 x}$$

$$v = c_2 e^{m_1 x}$$

From (3) $\quad (D - 1)y = c_2 e^{m_1 x}$

This is the linear differential equation.

$$\text{I.F.} = e^{-m_1 \int dx} = e^{-m_1 x}$$

Solution is

$$y.e^{-m_1 x} = \int (c_2 e^{m_1 x})(e^{-m_1 x})\, dx + c_1 = \int c_2\, dx + c_1 = c_2 x + c_1$$

$$y = (c_2 x + c_1) e^{m_1 x}$$

$$\text{C.F.} = (c_1 + c_2 x)\, e^{m_1 x}$$

Ex 1: Solve $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 15y = 0$

Solution. Given equation can be written as

$$(D^2 - 8D + 15)\,y = 0$$

Here auxiliary equation is $m^2 - 8m + 15 = 0$

$\Rightarrow \quad (m-3)(m-5) = 0 \qquad \therefore\ m = 3, 5$

Hence, the required solution is

$$y = C_1 e^{3x} + C_2 e^{5x}$$ **Ans.**

Ex 2: Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 0$

$$y = 2 \text{ and } \frac{dy}{dx} = \frac{d^2y}{dx^2} \text{ when } x = 0.$$

Solution. Here the auxiliary equation is

$$m^2 + 4m + 5 = 0$$

Its root are $-2 \pm i$

The complementary function is

$$y = e^{-2x}(A\cos x + B\sin x) \qquad \text{...(1)}$$

On putting $y = 2$ and $x = 0$ in (1), we get

$$2 = A$$

On putting $A = 2$ in (1), we have

$$y = e^{-2x}[2\cos x + B\sin x] \qquad \text{...(2)}$$

On differentiating (2), we get

$$\frac{dy}{dx} = e^{-2x}[-2\sin x + B\cos x] - 2e^{-2x}[2\cos x + B\sin x]$$

$$= e^{-2x}[(-2B-2)\sin x + (B-4)\cos x]$$

$$\frac{d^2y}{dx^2} = e^{-2x}[(-2B-2)\cos x - (B-4)\sin x]$$

$$-2e^{-2x}[(-2B-2)\sin x + (B-4)\cos x]$$

$$= e^{-2x}[(-4B+6)\cos x + (3B+8)\sin x]$$

But $\qquad \frac{dy}{dx} = \frac{d^2y}{dx^2}$

$$e^{-2x}[(-2B-2)\sin x + (B-4)\cos x] = e^{-2x}[(-4B+6)\cos x + (3B+8)\sin x]$$

On putting $x = 0$, we get

$$B - 4 = -4B + 6 \quad \Rightarrow \quad B = 2$$

(2) becomes, $\qquad y = e^{-2x}[2\cos x + 2\sin x]$

$$y = 2e^{-2x}[\sin x + \cos x]$$ **Ans.**

Exercise: Solve

1. $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0$ **Ans.** $y = (C_1 + C_2 x)\, e^{4x}$

2. $\frac{d^2y}{dx^2} + \mu^2 y = 0$ **Ans.** $y = C_1 \cos \mu x + C_2 \sin \mu x$

5.2 Rules for particular integral

(i) $\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$ If $f(a) = 0$ then $\frac{1}{f(D)} \cdot e^{ax} = x \cdot \frac{1}{f'(a)} \cdot e^{ax}$

If $f'(a) = 0$ then $\frac{1}{f(D)} \cdot e^{ax} = x^2 \frac{1}{f''(a)} \cdot e^{ax}$

(ii) $\frac{1}{f(D)} x^n = [f(D)]^{-1} x^n$ Expand $[f(D)]^{-1}$ and then operate.

(iii) $\frac{1}{f(D^2)} \sin ax = \frac{1}{f(-a^2)} \sin ax$ and $\frac{1}{f(D^2)} \cos ax = \frac{1}{f(-a^2)} \cos ax$

If $f(-a^2) = 0$ then $\frac{1}{f(D^2)} \sin ax = x \cdot \frac{1}{f'(-a^2)} \cdot \sin ax$

(iv) $\frac{1}{f(D)} e^{ax} \cdot \phi(x) = e^{ax} \cdot \frac{1}{f(D+a)} \phi(x)$

(v) $\frac{1}{D+a} \phi(x) = e^{-ax} \int e^{ax} \cdot \phi(x)\, dx$

$$\boxed{\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}.}$$

We know that, $D.e^{ax} = a.e^{ax}$, $D^2 e^{ax} = a^2.e^{ax}$, ..., $D^n e^{ax} = a^n e^{ax}$

Let $f(D)\, e^{ax} = (D^n + K_1 D^{n-1} + \ldots + K_n)\, e^{ax} = (a^n + K_1 a^{n-1} + \ldots + K_n) e^{ax} = f(a)\, e^{ax}$.

Operating both sides by $\frac{1}{f(D)}$

$$\frac{1}{f(D)} \cdot f(D) e^{ax} = \frac{1}{f(D)} \cdot f(a) e^{ax}$$

$$\Rightarrow \quad e^{ax} = f(a) \frac{1}{f(D)} \cdot e^{ax} \quad \Rightarrow \quad \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$$

If $f(a) = 0$, then the above rule fails.

Then $\frac{1}{f(D)} e^{ax} = x \cdot \frac{1}{f'(D)} e^{ax} = x \frac{1}{f'(a)} e^{ax}$ $\Rightarrow$ $\boxed{\frac{1}{f(D)} e^{ax} = x \,.\, \frac{1}{f'(a)} e^{ax}}$

If $f'(a) = 0$ then $\boxed{\frac{1}{f(D)} e^{ax} = x^2 \frac{1}{f''(a)} e^{ax}}$

Ex 3: Solve

$$\frac{d^2x}{dt^2}+\frac{g}{t}x=\frac{g}{l}L$$

where g, l, L are constants subject to the conditions,

$$x=a,\ \frac{dx}{dt}=0\ \text{ at } t=0.$$

Solution. We have, $\frac{d^2x}{dt^2}+\frac{g}{t}x=\frac{g}{l}L \quad \Rightarrow \quad \left(D^2+\frac{g}{l}\right)x=\frac{g}{l}L$

A.E. is $\quad m^2+\frac{g}{l}=0 \quad \Rightarrow \quad m=\pm\, i\sqrt{\frac{g}{l}}$

$$\text{C.F.} = C_1\cos\sqrt{\frac{g}{l}}\,t+C_2\sin\sqrt{\frac{g}{l}}\,t$$

$$\text{P.I.} = \frac{1}{D^2+\frac{g}{l}}\cdot\frac{g}{l}L=\frac{g}{l}L\frac{1}{D^2+\frac{g}{l}}e^{0t}=\frac{g}{l}L\frac{1}{0+\frac{g}{l}}=L \qquad [D=0]$$

$\therefore$ General solution is = C.F. + P.I.

$$x=C_1\cos\left(\sqrt{\frac{g}{l}}\right)t+C_2\sin\left(\sqrt{\frac{g}{l}}\right)t+L \qquad ...(1)$$

$$\frac{dx}{dt}=-C_1\sqrt{\frac{g}{l}}\sin\left(\sqrt{\frac{g}{l}}\right)t+C_2\sqrt{\frac{g}{l}}\cos\left(\sqrt{\frac{g}{l}}\right)t$$

Put $t=0$ and $\frac{dx}{dt}=0$

$0=C_2\sqrt{\frac{g}{l}} \qquad \therefore \quad C_2=0$

(1) becomes $\quad x=C_1\cos\sqrt{\frac{g}{l}}\,t+L \qquad ...(2)$

Put $x=a$ and $t=0$ in (2), we get

$a=C_1+L \quad$ or $\quad C_1=a-L$

On putting the value of C_1 in (2), we get $x=(a-L)\cos\left(\sqrt{\frac{g}{l}}\right)t+L$ **Ans.**

Ex 4: Solve

$$\frac{d^2y}{dx^2}+6\frac{dy}{dx}+9y=5e^{3x}$$

Solution. $(D^2 + 6D + 9)y = 5e^{3x}$

Auxiliary equation is $m^2 + 6m + 9 = 0 \quad \Rightarrow \quad (m + 3)^2 = 0 \quad \Rightarrow \quad m = -3, -3,$

$$\text{C.F.} = (C_1 + C_2x)\, e^{-3x}$$

$$\text{P.I.} = \frac{1}{D^2+6D+9}.5.e^{3x} = 5\frac{e^{3x}}{(3)^2+6(3)+9} = \frac{5e^{3x}}{36}$$

The complete solution is $\quad y = (C_1 + C_2x)e^{-3x} + \dfrac{5e^{3x}}{36}$ **Ans.**

Ex 5: ***Solve***

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 6e^{3x} + 7e^{-2x} - \log 2$$

Solution. $(D^2 - 6D + 9)y = 6e^{3x} + 7e^{-2x} - \log 2$

A.E. is $(m^2 - 6m + 9) = 0 \quad \Rightarrow \quad (m - 3)^2 = 0, \quad \Rightarrow \quad m = 3, 3$

$$\text{C.F.} = (C_1 + C_2x)e^{3x}$$

$$\text{P.I.} = \frac{1}{D^2-6D+9}6e^{3x} + \frac{1}{D^2-6D+9}7e^{-2x} + \frac{1}{D^2-6D+9}(-\log 2)$$

$$= x\frac{1}{2D-6}6e^{3x} + \frac{1}{4+12+9}7e^{-2x} - \log 2\frac{1}{D^2-6D+9}e^{0x}$$

$$= x^2\frac{1}{2}\cdot 6\cdot e^{3x} + \frac{7}{25}e^{-2x} - \log 2\left(\frac{1}{9}\right) = 3x^2e^{3x} + \frac{7}{25}e^{-2x} - \frac{1}{9}\log 2$$

Complete solution is $\; y = (C_1 + C_2\, x)e^{3x} + 3x^2e^{3x} + \dfrac{7}{25}e^{-2x} - \dfrac{1}{9}\log 2$ **Ans.**

Exercise

Solve

1. $[D^2 + 5D + 6]\,[y] = e^x$ — **Ans.** $C_2\, e^{-2x} + C_2e^{-3x} + \dfrac{e^x}{12}$

2. $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = e^{3x}$ — **Ans.** $C_1e^x + C_2e^{2x} + \dfrac{e^{3x}}{2}$
   (A.M.I.E.T.E. June 2010, 2007)

3. $(D^3 + 2D^2 - D - 2)\, y = e^x$ — **Ans.** $C_1e^x + C_2e^{-x} + C_3e^{-2x} + \dfrac{x}{6}e^x$

4. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 2y = \sinh x$ — **Ans.** $e^{-x}[C_1\cos x + C_2\sin x] + \dfrac{e^x}{10} - \dfrac{e^{-x}}{2}$

5. $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 5y = -2\cosh x$ — **Ans.** $e^{-2x}(C_1\cos x + C_2\sin x) - \dfrac{1}{10}e^x - \dfrac{e^{-x}}{2}$

$$\boxed{\frac{1}{f(D)} x^n = [f(D)]^{-1} x^n.}$$

Ex 6 : ***Solve***

$$\frac{d^2y}{dx^2} + a^2 y = \frac{a^2 R}{p}(l - x)$$

where a, R, p and l are constants subject to the conditions $y = 0,\ \frac{dy}{dx} = 0$ *at* $x = 0$.

Solution. $\frac{d^2y}{dx^2} + a^2 y = \frac{a^2}{p} R(l - x) \quad \Rightarrow \quad (D^2 + a^2)y = \frac{a^2}{p} R(l - x)$

A.E. is $m^2 + a^2 = 0 \Rightarrow m = \pm ia$

C.F. $= C_1 \cos ax + C_2 \sin ax$

$$\text{P.I.} = \frac{1}{D^2 + a^2} \frac{a^2}{p} R(l - x) = \frac{a^2 R}{p} \frac{1}{a^2} \left[\frac{1}{1 + \frac{D^2}{a^2}} \right] (l - x) = \frac{R}{p} \left[1 + \frac{D^2}{a^2} \right]^{-1} (l - x)$$

$$= \frac{R}{p} \left[1 - \frac{D^2}{a^2} \right] (l - x) = \frac{R}{p}(l - x)$$

$$y = C_1 \cos ax + C_2 \sin ax + \frac{R}{p}(l - x) \qquad \text{...(1)}$$

On differentiating (1), we get $\frac{dy}{dx} = -a\, C_1 \sin ax + a\, C_2 \cos ax - \frac{R}{p}$ ...(2)

On putting $\frac{dy}{dx} = 0$ and $x = 0$ in (2), we have

$$0 = a\, C_2 - \frac{R}{p} \quad \Rightarrow \quad C_2 = \frac{R}{a.p}$$

On putting the values of C_1 and C_2 in (1), we get

$$y = -\frac{R}{p} l \cos a.x + \frac{R}{a.p} \sin a.x + \frac{R}{p}(l - x) \Rightarrow y = \frac{R}{p}\left[\frac{\sin ax}{a} - l \cos a.x + l - x \right] \qquad \textbf{Ans.}$$

Exercise

Solve

1. $(D^2 + 5D + 4)\, y = 3 - 2x$ **Ans.** $C_1 e^{-x} + C_2 e^{-4x} + \frac{1}{8}(11 - 4x)$
2. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x$ **Ans.** $(C_1 + C_2 x)\, e^{-x} + x - 2$
3. $(2D^2 + 3D + 4)\, y = x^2 - 2x$ **Ans.** $e^{-\frac{3}{4}x}[A \cos \frac{\sqrt{23}}{4} x + B \sin \frac{\sqrt{23}}{4} x] + \frac{1}{32}[8x^2 - 28x + 13]$

$$\boxed{\frac{1}{f(D^2)}\sin ax = \frac{\sin ax}{f(-a^2)}}$$

$$\boxed{\frac{1}{f(D^2)}\cdot\cos ax = \frac{\cos ax}{f(-a^2)}}$$

$D(\sin ax) = a.\cos ax$, $D^2(\sin ax) = D(a\cos ax) = -a^2.\sin ax$

$D^4(\sin ax) = D^2.D^2(\sin ax) = D^2(-a^2\sin ax) = (-a^2)^2\sin ax$

$(D^2)^n\sin ax = (-a^2)^n\sin ax$

Hence, $f(D^2)\sin ax = f(-a^2)\sin ax$

$$\frac{1}{f(D^2)}\cdot f(D^2)\sin ax = \frac{1}{f(D^2)}.f(-a^2).\sin ax$$

$$\sin ax = f(-a^2)\frac{1}{f(D^2)}\sin ax \quad\Rightarrow\quad \frac{1}{f(D^2)}\cdot\sin ax = \frac{\sin ax}{f(-a^2)}$$

Similarly, $$\frac{1}{f(D^2)}\cos ax = \frac{\cos ax}{f(-a^2)}$$

If $f(-a^2) = 0$ then above rule fails.

$$\frac{1}{f(D^2)}\sin ax = x\frac{\sin ax}{f'(-a^2)}$$

If $f'(-a^2) = 0$ then, $$\frac{1}{f(D^2)}\sin ax = x^2\frac{\sin ax}{f''(-a^2)}$$

Ex 7: Solve

$(D^2 + 4)\,y = \cos 2x$

Solution. $(D^2 + 4)\,y = \cos 2x$

Auxiliary equation is $m^2 + 4 = 0$

$m = \pm 2i$, C.F. $= A\cos 2x + B\sin 2x$

$$\text{P.I.} = \frac{1}{D^2+4}\cos 2x = x.\frac{1}{2D}\cos 2x = \frac{x}{2}\left(\frac{1}{2}\sin 2x\right) = \frac{x}{4}\sin 2x$$

Complete solution is $y = A\cos 2x + B\sin 2x + \frac{x}{4}\sin 2x$ **Ans.**

Ex 8: Solve

$$\frac{d^3y}{dx^3}-3\frac{d^2y}{dx^3}+4\frac{dy}{dx}-2y=e^x+\cos x$$

Solution. Given $(D^3 - 3D^2 + 4D - 2)\, y = e^x + \cos x$

A.E. is $m^3 - 3m^2 + 4m - 2 = 0$

$\Rightarrow \quad (m - 1)(m^2 - 2m + 2) = 0$, *i.e.*, $m = 1,\ 1 \pm i$

$\therefore \quad$ C.F. $= C_1e^x + e^x (C_2 \cos x + C_3 \sin x)$

$$\text{P.I.} = \frac{1}{(D-1)(D^2-2D+2)}e^x + \frac{1}{D^3-3D^2+4D-2}\cos x$$

$$= \frac{1}{(D-1)(1-2+2)}e^x + \frac{1}{(-1)D-3(-1)+4D-2}\cos x$$

$$= \frac{1}{(D-1)}e^x + \frac{1}{3D+1}\cos x = x\frac{1}{1}e^x + \frac{3D-1}{9D^2-1}\cos x$$

$$= e^x.x + \frac{(-3\sin x - \cos x)}{-9-1} = e^x.x + \frac{1}{10}(3\sin x + \cos x)$$

Hence, complete solution is

$$y = C_1e^x + e^x(C_2\cos x + C_3\sin x) + x\,e^x + \frac{1}{10}(3\sin x + \cos x) \qquad \textbf{Ans.}$$

Ex 9: Solve

Solution. $\qquad (D^3+1)y = \cos^2\left(\frac{x}{2}\right) + e^{-x}$

A.E. is $m^3 + 1 = 0$

$$(m + 1)(m^2 - m + 1) = 0 \quad \Rightarrow \quad m = -1$$

or $\qquad m = \frac{-(-1)\pm\sqrt{1-4}}{2} = \frac{1\pm i\sqrt{3}}{2} \quad \Rightarrow \quad m = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

$$\therefore \quad \text{C.F.} = C_1e^{-x} + e^{\frac{x}{2}}\left[C_2\cos\frac{\sqrt{3}}{2}x + C_3\sin\frac{\sqrt{3}}{2}x\right]$$

$$\text{P.I.} = \frac{1}{D^3+1}\left[\cos^2\left(\frac{x}{2}\right) + e^{-x}\right] = \frac{1}{D^3+1}\cos^2\left(\frac{x}{2}\right) + \frac{1}{D^3+1}e^{-x} \qquad [\text{Put } D = -1]$$

$$= \frac{1}{D^3+1}\left(\frac{1+\cos x}{2}\right) + \frac{1}{3D^2+1}e^{-x}$$

$$= \frac{1}{2}\frac{1}{D^3+1}e^{0x} + \frac{1}{2}\frac{1}{D^3+1}\cos x + \frac{1}{3(-1)^2+1}e^{-x} = \frac{1}{2} + \frac{1}{2}\frac{1}{-D+1}\cos x + \frac{1}{4}e^{-x}$$

$$= \frac{1}{2} - \frac{1}{2}\frac{(D+1)\cos x}{(D-1)(D+1)} + \frac{1}{4}e^{-x} = \frac{1}{2} - \frac{1}{2}\frac{(-\sin x + \cos x)}{(D^2-1)} + \frac{1}{4}e^{-x}$$

$$(D^3+1)y = \cos^2\left(\frac{x}{2}\right) + e^{-x}$$

$$= \frac{1}{2} + \frac{1}{2}\frac{\sin x}{(D^2-1)} - \frac{1}{2}\frac{1}{(D^2-1)}\cos x + \frac{1}{4}e^{-x}$$

Put $D^2 = -1 = \frac{1}{2} + \frac{1}{2}\frac{\sin x}{(-1-1)} - \frac{1}{2}\frac{1}{(-1-1)}\cos x + \frac{1}{4}e^{-x} = \frac{1}{2} - \frac{\sin x}{4} + \frac{\cos x}{4} + \frac{1}{4}e^{-x}$

$$\text{P.I.} = \frac{1}{2} + \frac{1}{4}(\cos x - \sin x + e^{-x})$$

Hence, the complete solution is

$$y = C_1 e^{-x} + e^{\frac{x}{2}}\left[C_2 \cos\frac{\sqrt{3}}{2}x + C_3 \sin\frac{\sqrt{3}}{2}x\right] + \frac{1}{2} + \frac{1}{4}(\cos x - \sin x + e^{-x})$$ **Ans.**

5.3 Summary

Within a particular class of ordinary differential equations (ODEs), known as homogeneous differential equations, every term may be represented as a function of the dependent variable and its derivatives. When modeling systems where all terms may be represented as homogeneous functions of the dependent variable and its derivatives, homogeneous differential equations offer a useful foundation. Their answers provide understanding of these systems' behavior and are crucial resources for mathematical modeling and research.

5.4 Keywords

- Differential equation
- Homogeneous differential equation
- Complementry function
- Particular inegral

5.5 Self Assessment Questions

1. $\frac{d^2y}{dx^2} + 6y = \sin 4x$ **Ans.** $C_1 \cos\sqrt{6}x + C_2 \sin\sqrt{6}x - \frac{1}{10}\sin 4x$

2. $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 3x = \sin t$ **Ans.** $e^{-t}[A\cos\sqrt{2}t + B\sin\sqrt{2}t] - \frac{1}{4}(\cos t - \sin t)$

3. $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = \sin 2t$, given that when $t = 0$, $x = 3$ and $\frac{dx}{dt} = 0$

Ans. $e^{-t}\left[\frac{55}{17}\cos 2t + \frac{53}{34}\sin 2t\right] - \frac{1}{17}(4\cos 2t - \sin 2t)$

5.6 Case Study

An analysis of electrical circuits:

When conventional techniques become unfeasible owing to circuit complexity or nonlinearity, series solutions can be utilized in electrical engineering to examine intricate circuits.

Question: Learn about the behavior and functionality of a complicated electrical circuit by analyzing it.

5.7 References

1. "Elementary Differential Equations and Boundary Value Problems" by William E. Boyce and Richard C. DiPrima
2. "Advanced Engineering Mathematics" by Erwin Kreyszig

UNIT – 6
Numerical Solution of Differential Equations

Learning objectives

- Give an explanation of differential equations and their use in dynamic system modeling.
- Identify the distinctions between partial differential equations (PDEs) and ordinary differential equations (ODEs).
- Explain that numerical solutions to differential equations are approximations produced by computer techniques.

Structure

6.1 Euler's Method

The foundation of Euler's approach is the notion that the value of the solution at the next point may be estimated using the tangent line at the present position. With this approach, the time domain is discretized into tiny steps, and the solution is advanced by looking at the slope of the solution curve.

Here's a basic outline of how Euler's method works:

1. **Start with an initial condition**: You begin with an initial value for the dependent variable (usually denoted as y0) at a given point in time (usually denoted as t0).
2. **Choose a step size**: Determine the size of the time steps (Δt) that you will use to discretize the interval over which you want to approximate the solution.

   - **Iterate using Euler's method**
   - **Calculate the derivative**: Evaluate the derivative of the function at the current point.
   - **Update the function value**: Multiply the derivative by the step size and add it to the current function value to obtain the next function value.
   - **Update the time**: Move to the next time step by adding the step size to the current time.

3. **Repeat until you reach the desired endpoint**: Continue this process until you reach the desired endpoint or until the desired accuracy is achieved.

$y - y_0 = m(x - x_0)$, where m is slope of tangent at the point (x_0, y_0)

Also $m = \left.\frac{dy}{dx}\right]_{(x_0,y_0)} = f(x_0, y_0)$ from ①

$\Rightarrow y = y_0 + f(x_0, y_0)\,(x - x_0)$

$\Rightarrow y_1 = y_0 + f(x_0, y_0)\,(x_1 - x_0) \qquad \because y(x_1) = y_1$

$\Rightarrow y_1 = y_0 + hf(x_0, y_0) \qquad \because x_1 - x_0 = h$

Similarly for range $[x_1, x_2]$

$y_2 = y_1 + hf(x_1, y_1)$

$\vdots$

$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$

Example 1:
Using Euler's method , compute y(0.12)

$\frac{dy}{dx} = x^3 + y$; $y(0) = 1$, taking $h = 0.02$.

Solution: Given $f(x,y) = x^3 + y$, $x_0 = 0$, $y_0 = 1$, $x_n = x_{n-1} + h$, $h = 0.02$

$\therefore x_1 = 0.02$, $x_2 = 0.04$, $x_3 = 0.06$, $x_4 = 0.08$, $x_5 = 0.1$

Using Euler's method $y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$

$\Rightarrow \; y_n = y_{n-1} + h({x_{n-1}}^3 + y_{n-1}) \qquad \cdots ①$

Putting $n = 1$ in ①, $y_1 = y(0.02) = y_0 + h\,({x_0}^3 + y_0)$

$\therefore \; y_1 = 1 + 0.02(0 + 1) = 1.02$

Putting $n = 2$ in ①, $y_2 = y(0.04) = y_1 + h\,({x_1}^3 + y_1)$

$\therefore \; y_2 = 1.02 + 0.02((0.02)^3 + 1.02) = 1.04040016$

Putting $n = 3$ in ①, $y_3 = y(0.06) = y_2 + h\,({x_2}^3 + y_2)$

$\therefore \; y_3 = 1.04040016 + 0.02((0.04)^3 + 1.04040016) = 1.061209443$

Putting $n = 4$ in ①, $y_4 = y(0.08) = y_3 + h\,({x_3}^3 + y_3)$

$\therefore \; y_4 = 1.061209443 + 0.02((0.06)^3 + 1.061209443) = 1.082437952$

Putting $n = 5$ in ①, $y_5 = y(0.1) = y_4 + h\,({x_4}^3 + y_4)$

$\therefore \; y_5 = 1.082437952 + 0.02((0.08)^3 + 1.082437952) = 1.104096951$

Putting $n = 6$ in ①, $y_6 = y(0.12) = y_5 + h\,({x_5}^3 + y_5)$

$\therefore \; y_6 = 1.104096951 + 0.02((0.1)^3 + 1.104096951) = 1.126198890$

Thus at $x = 0.12$, $y = 1.126198890 \;\Rightarrow y(0.12) = 1.126198890$

6.2 Modified Euler's Method:

$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$

$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$

$\vdots$

Continue approximating y_1 until two consecutive values are coincident to a specific degree of accuracy.

$\therefore y_1^{(k)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(k-1)})]$

Repeat the procedure for y_2 , y_3, y_4 ... to find y_n

Example 2:
Using Modified Euler's Method,compute y(0.2),y(0.4)

$$\frac{dy}{dx} = y - x^2 \; ; \; y(0) = 1$$

Solution: Given $f(x, y) = y - x^2$, $x_0 = 0$, $y_0 = 1$

By Euler's method $y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$

$i.$ To evaluate $y(0.2)$, $h = 0.2$, $x_1 = 0 + 0.2 = 0.2$

$y_1 = y(0.2) = y_0 + hf(x_0, y_0)$, $f(x_0, y_0) = y_0 - x_0^2 = 1 - 0 = 1$

$\therefore \; y_1 = 1 + 0.2(1) = 1.2$

$f(x_1, y_1) = y_1 - x_1^2 = 1.2 - (0.2)^2 = 1.16$

Now improving y_1 using Modified Euler's method

$$y_1^{(1)} = y_0 + \frac{h}{2}(f(x_0, y_0) + \; f\,(x_1\,, y_1))$$

$$\therefore y_1^{(1)} = 1 + \frac{0.2}{2}(1 + 1.16) = 1.216$$

$$f(x_1\,, y_1^{(1)}) = y_1^{(1)} - x_1^2 = 1.216 - (0.2)^2 = 1.176$$

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + \; f(x_1\,, y_1^{(1)})]$$

$$\therefore y_1^{(2)} = 1 + \frac{0.2}{2}(1 + 1.176) = 1.2176$$

$$f(x_1\,, y_1^{(2)}) = y_1^{(2)} - x_1^2 = 1.2176 - (0.2)^2 = 1.1776$$

$$y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + \; f(x_1\,, y_1^{(2)})]$$

$$\therefore y_1^{(3)} = 1 + \frac{0.2}{2}(1 + 1.1776) = 1.21776 = y(0.2)$$

Thus by Modified Euler's method, we have improved $y(0.2)$ from 1.2 to 1.21776

$ii.$ To evaluate $y(0.4)$, $h = 0.2$, $x_2 = 0.2 + 0.2 = 0.4$

$$y_2 = y(0.4) = y_1 + hf(x_1, y_1),$$

$$f(x_1, y_1) = y_1 - x_1^2 = 1.21776 - (0.2)^2 = 1.17776$$

$$\therefore \; y_2 = 1.21776 + 0.2(1.17776) = 1.453312$$

$$f(x_2, y_2) = y_2 - x_2^2 = 1.453312 \; - (0.4)^2 = 1.293312$$

Now improving y_1 using Modified Euler's method

$$y_2^{(1)} = y_1 + \frac{h}{2}(f(x_1, y_1) + f(x_2, y_2))$$

$$\therefore y_2^{(1)} = 1.21776 + \frac{0.2}{2}(1.17776 + 1.293312) = 1.4648672$$

$$f\left(x_2, y_2^{(1)}\right) = y_2^{(1)} - x_2^2 = 1.4648672 - (0.4)^2 = 1.3048672$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left(f(x_1, y_1) + f\left(x_2, y_2^{(1)}\right)\right)$$

$$\therefore y_2^{(2)} = 1.21776 + \frac{0.2}{2}(1.17776 + 1.3048672) = 1.46602272$$

$$f\left(x_2, y_2^{(2)}\right) = y_2^{(2)} - x_2^2 = 1.46602272 - (0.4)^2 = 1.30602272$$

$$y_2^{(3)} = y_1 + \frac{h}{2}\left(f(x_1, y_1) + f\left(x_2, y_2^{(2)}\right)\right)$$

$$\therefore y_2^{(3)} = 1.21776 + \frac{0.2}{2}(1.17776 + 1.30602272) = 1.466138272$$

Thus by Modified Euler's method, we have improved $y(0.4)$ from 1.453312 to 1.466138272 correct to 3 decimal places.

Example 3:

Using Modified Euler's Method,compute y(1.2).

$$\frac{dy}{dx} = \ln(x + y)\ ;\ y(1) = 2$$

Solution: Given $f(x, y) = \ln(x + y)$, $x_0 = 1$, $y_0 = 2$

By Euler's method $y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$

To evaluate $y(1.2)$, $h = 0.2, x_1 = 1 + 0.2 = 1.2$

$$y_1 = y(1.2) = y_0 + hf(x_0, y_0)$$

$$f(x_0, y_0) = ln\,(x_0 + y_0) = \ln(1 + 2) = 1.09861$$

$$\therefore y_1 = 2 + 0.2(1.09861) = 2.21972$$

$$f(x_1, y_1) = ln(x_1 + y_1) = \ln(1 + 2.21972) = 1.16929$$

$$y_1^{(1)} = y_0 + \frac{h}{2}(f(x_0, y_0) + f(x_1, y_1))$$

$$\therefore y_1^{(1)} = 2 + \frac{0.2}{2}(1.09861 + 1.16929) = 2.22679$$

$$f\left(x_1, y_1^{(1)}\right) = \ln\left(x_1 + y_1^{(1)}\right) = \ln(1 + 2.22679) = 1.17149$$

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)]$$

$$\therefore y_1^{(2)} = 2 + \frac{0.2}{2}(1.09861 + 1.17149) = 2.22701$$

$$f\left(x_1, y_1^{(2)}\right) = \ln\left(x_1 + y_1^{(2)}\right) = \ln(1 + 2.22701) = 1.17156$$

$$y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f\left(x_1, y_1^{(2)}\right)]$$

$$\therefore y_1^{(3)} = 2 + \frac{0.2}{2}(1.09861 + 1.17156) = 2.227017 = y(1.2)$$

6.3 Picard's Method:

Consider the initial value problem given by $\frac{dy}{dx} = f(x, y)$; $y(x_0) = y_0$

$\Rightarrow dy = f(x, y)dx$ Integrating, we get

$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)dx$

$\Rightarrow y - y_0 = \int_{x_0}^{x} f(x, y)dx$

$\Rightarrow y = y_0 + \int_{x_0}^{x} f(x, y)dx$

To obtain the first approximation, replacing y by y_0 on R.H.S.

$$\Rightarrow \quad y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)dx$$

Similarly $y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)dx$

$$\vdots$$

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})dx \quad \text{, where } y(x_0) = y_0$$

Example 4:

Use Picard's Method , solve the IVP $\frac{dy}{dx} = x + y$, $y(0) = 1$

Solution: Given $f(x, y) = x + y$, $\quad x_0 = 0$, $y_0 = 1$

Using Picard's approximation

$$y = y_0 + \int_{x_0}^{x} f(x, y)dx$$

1st approximation:

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)dx$$

$$\Rightarrow y_1 = 1 + \int_{0}^{x} (x + 1)dx$$

$$= 1 + \left[\frac{x^2}{2} + x\right]_0^x = 1 + x + \frac{x^2}{2}$$

2nd approximation:

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)dx$$

$$\Rightarrow y_2 = 1 + \int_{0}^{x} (x + y_1)dx$$

$$= 1 + \int_{0}^{x} \left(x + \left(1 + x + \frac{x^2}{2}\right)\right) dx$$

$$= 1 + x + x^2 + \frac{x^3}{6}$$

3rd approximation:

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)dx$$

$$\Rightarrow \; y_3 = 1 + \int_0^x (x + y_2)dx$$

$$= 1 + \int_0^x \left(x + \left(1 + x + x^2 + \frac{x^3}{6}\right)\right) dx$$

$$= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}$$

Example 5:
Use Picard's Method , solve the IVP
$\frac{dy}{dx} = x(1 + x^3 y)$;

Solution: Given $f(x, y) = x(1 + x^3 y)$, $\quad x_0 = 0, \; y_0 = 3$

Using Picard's approximation

$$y = y_0 + \int_{x_0}^{x} f(x, y)dx$$

1st approximation:

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)dx$$

$$\Rightarrow \; y_1 = 3 + \int_0^x x(1 + x^3 y) \, dx$$

$$= 3 + \frac{x^2}{2} + \frac{3x^5}{5}$$

2nd approximation:

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)dx$$

$$\Rightarrow \; y_2 = 3 + \int_0^x x\left[1 + x^3\left(3 + \frac{x^2}{2} + \frac{3x^5}{5}\right)\right] dx$$

$$= 3 + \frac{x^2}{2} + \frac{3x^5}{5} + \frac{x^7}{14} + \frac{3x^{10}}{50}$$

Clearly y_1 and y_2 are coincident upto 3 terms.

$\therefore$ Let $y = 3 + \frac{x^2}{2} + \frac{3x^5}{5}$

Also $y(0.1) = 3 + \frac{(0.1)^2}{2} + \frac{3(0.1)^5}{5} = 3.00501$

6.4 Taylor's Method:

Taylor's series expansion of a function $y(x)$ about $x = x_0$ is given by

$$y(x) = y_0 + (x - x_0)y_0' + \frac{1}{2!}(x - x_0)^2 y_0'' + \frac{1}{3!}(x - x_0)^3 y_0''' + \cdots \qquad \cdots ①$$

Example 6: Solve the D.E

$\frac{dy}{dx} = x + y$; $y(0) = 1$, at $x = 0.2$,

Solution: Taylor's series expansion of $y(x)$ about $x = 0$ is given by:

$$y(x) = y_0 + (x-0)y_0' + \frac{1}{2!}(x-0)^2 y_0'' + \frac{1}{3!}(x-0)^3 y_0''' + \frac{1}{4!}(x-0)^4 y_0^{iv} + \cdots \quad \cdots ①$$

$$\begin{aligned}
\text{Given} \quad & \frac{dy}{dx} = x + y \; ; \quad y_0 = 1 \\
\text{or} \quad & y' = x + y \; ; \quad y_0' = 1 \\
\Rightarrow \quad & y'' = 1 + y' \; ; \quad y_0'' = 2 \\
& y''' = y'' \; ; \quad y_0''' = 2 \\
& y^{iv} = y''' \; ; \quad y_0^{iv} = 2 \\
& \vdots
\end{aligned}$$

Substituting these values in ①, we get

$$y(x) = 1 + x(1) + \frac{1}{2!}x^2(2) + \frac{1}{3!}x^3(2) + \frac{1}{4!}x^4(2) + \cdots$$

Or $\quad y(x) = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \cdots$

$i.\quad y(0.2) = 1 + 0.2 + 0.04 + \frac{0.008}{3} + \frac{0.0016}{12} + \cdots$

$\qquad = 1 + 0.2 + 0.04 + 0.002667 + 0.00013 + \cdots$

The fifth term in this series is $0.00013 < 0.0005$

Hence value of $y(0.2)$ correct to 3 decimal places may be obtained by adding first four terms.

$\therefore y(0.2) \approx 1.24280 \approx 1.243$

$ii.\quad y(0.4) = 1 + 0.4 + 0.16 + \frac{0.064}{3} + \frac{0.0256}{12} + \frac{0.01024}{60} + \cdots$

$\qquad = 1 + 0.4 + 0.16 + 0.02133 + 0.00213 + 0.00017 + \cdots$

The sixth term in this series is $0.00017 < 0.0005$

Therefore, by summing the first five terms, the value of y (0.4), accurate to three decimal places, may be determined. y(0.4) is accurate to three decimal places: 1.58346 = 1.583.

Again to find exact solution of $\frac{dy}{dx} - y = x$, which is a linear differential equation

Integrating Factor (I.F.) $= e^{\int -dx} = e^{-x}$

Solution is given by $ye^{-x} = \int xe^{-x}dx$

$$\Rightarrow \quad ye^{-x} = -xe^{-x} - e^{-x} + c$$

$$\Rightarrow \quad y = -x - 1 + ce^{x}$$

Given that $y(0) = 1 \quad \Rightarrow 1 = 0 - 1 + c \qquad \therefore c = 2$

$\Rightarrow y = -x - 1 + 2e^{x}$

$y(0.2) \approx 1.243$ and $y(0.4) \approx 1.584$ correct to three decimal places

6.5 Runge-Kutta Method:

The concepts employed in the Euler and Modified Euler methods are preferred in the Runge-Kutta method.

Think about the original value issue.

$$\frac{dy}{dx} = f(x,y)\ ; \quad y(x_0) = y_0 \qquad \cdots ①$$

The formula for the Taylor's series expansion of a function y(x) about x = x1 is

$$y(x) = y_0 + (x - x_0)y_0' + \frac{1}{2!}(x - x_0)^2 y_0'' + \frac{1}{3!}(x - x_0)^3 y_0''' + \cdots$$

Now $y_1 = y(x_0 + h)$, $\therefore$ Putting $x = x_0 + h$ in Taylor's series, we get

$$y_1 = y(x_0 + h) = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \cdots \qquad \cdots ②$$

Also by Euler's method $y_1 = y_0 + hf(x_0, y_0) = y_0 + hy_0' \qquad \cdots ③$

From (2) and (3), Taylor's series expansion is consistent with Euler's technique up to the first two terms, or the term that contains the h of order one.

The Runge-Kutta method is first order in Euler's approach.

Modified Euler's method is given by $y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$

$\Rightarrow \quad y_1 = y_0 + \frac{1}{2}[hf(x_0, y_0) + hf(x_1, y_1)]$

Now $x_1 = x_0 + h$ and $y_1 = y_0 + hf(x_0, y_0)$ by Euler's method

$\Rightarrow \quad y_1 = y_0 + \frac{1}{2}[hf(x_0, y_0) + hf\big(x_0 + h, y_0 + hf(x_0, y_0)\big)]$

$\Rightarrow y_1 = y_0 + \frac{1}{2}[K_1 + K_2]$

Where $K_1 = hf(x_0, y_0)$, $K_2 = hf(x_0 + h, y_0 + K_1)$

Therefore , Modified Euler's method itself is second order Runge-Kutta method.

Similarly, up to the first four terms, or until the term containing order three term h, which is given by, the third order Runge-Kutta method agrees with Taylor's series expansion.

$$y_1 = y_0 + \frac{1}{6}[K_1 + 4K_2 + K_3]$$
$$\text{where } K_1 = hf(x_0, y_0)$$
$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right),$$
$$K_3 = hf(x_0 + h, y_0 + hf(x_0 + h, y_0 + K_1)$$

In a similar vein, Taylor's series expansion is ancillary with Runge-Kutta's approach of order four up to the first five terms, or until the term containing order four h. This is the numerical solution for D's initial value problem using the fourth order Runge-Kutta method:

$$y_1 = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$$
$$\text{where } K_1 = hf(x_0, y_0)$$
$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$$
$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$$
$$K_4 = hf(x_0 + h, y_0 + K_3)$$

Example 7:

Solve the differential equation $\frac{dy}{dx} = y - x$; $y(0) = 1$, at $x = 0.1$, using Runge-Kutta method. Also compare the numerical solution obtained with the exact solution.

Solution: Given $f(x, y) = x + y$, $x_0 = 0$, $y_0 = 1$, $h = 0.1$

Runge-Kutta method of 4th order is given by

$$y_1 = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4] \quad \cdots ①$$

$$K_1 = hf(x_0, y_0) = h(y_0 - x_0) = 0.1(1 - 0) = 0.1$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = 0.1\left(\left(1 + \frac{0.1}{2}\right) - \left(0 + \frac{0.1}{2}\right)\right) = 0.1$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = 0.1\left(\left(1 + \frac{0.1}{2}\right) - \left(0 + \frac{0.1}{2}\right)\right) = 0.1$$

$$K_4 = hf(x_0 + h, y_0 + K_3) = 0.1\big((1 + 0.1) - (0 + 0.1)\big) = 0.1$$

Substituting values of K_1, K_2, K_3, K_4 in ①, we get the solution as:

$$y_1 = 1 + \frac{1}{6}[0.1 + 2(0.1) + 2(0.1) + 0.1] = 1.1$$

Again to find exact solution of the initial value problem

$\frac{dy}{dx} - y = -x$, which is a linear differential equation

Integrating Factor (I.F.) $= e^{\int -dx} = e^{-x}$

Solution is given by $ye^{-x} = -\int xe^{-x}dx$

$$\Rightarrow \quad ye^{-x} = xe^{-x} + e^{-x} + c$$

$$\Rightarrow \quad y = x + 1 + ce^{x}$$

Given that $y(0) = 1 \quad \Rightarrow 1 = 0 + 1 + c \qquad \therefore c = 0$

$$\Rightarrow y = x + 1$$

$$y(0.1) = 0.1 + 1 = 1.1$$

6.6 Shooting Method

The shooting method is a numerical technique used to solve BVPs. Unlike initial value problems (IVPs), where the solution is specified at a single point, BVPs require that the solution satisfies conditions at both the initial and final points (or at multiple points).

Here's how the shooting method typically works:

1. **Formulate the problem**: Write down the differential equation(s) and specify the boundary conditions at the endpoints.
2. **Convert the BVP to an IVP**: Transform the BVP into an equivalent initial value problem by guessing values for the unknown boundary condition(s) at one endpoint.

3. **Integrate the IVP**: Use a numerical method like Euler's method, Runge-Kutta methods, or a more advanced technique to integrate the transformed initial value problem from the initial point to the final point.
4. **Adjust the guessed boundary condition(s)**: Compare the value(s) of the dependent variable obtained at the final point of integration with the desired boundary condition(s) at that point. Adjust the guessed boundary condition(s) until the solution satisfies the desired boundary conditions at the final point.
5. **Iterate**: Repeat steps 3 and 4 until the solution obtained satisfies the boundary conditions at the final point within the desired tolerance.

The name "shooting method" comes from the analogy of shooting a target: you "shoot" an initial guess for the boundary condition(s) and adjust it until you hit the target (i.e., satisfy the boundary conditions at the final point).

The shooting method is widely used for solving BVPs, especially when direct methods like finite difference methods are not applicable or when coupled with other techniques like finite element methods. It's versatile and applicable to a wide range of problems, but it can sometimes be computationally expensive, especially if the initial guess requires many iterations to converge to the correct solution.

6.7 Summary

In summary, numerical solutions of differential equations play a vital role in scientific and engineering endeavors, offering practical tools for modeling and simulating dynamic systems and phenomena where analytical solutions are challenging or impossible to obtain.

6.8 Keywords

- Numerical Solution
- Differential Equations
- Ordinary Differential Equations (ODEs)
- Partial Differential Equations (PDEs)
- Euler's Method

6.9 Self-Assessment questions

1. What is the purpose of numerical solution of differential equations?
2. What are the main types of differential equations solved numerically?
3. What are the advantages of higher-order numerical methods over Euler's method?
4. How do adaptive step size methods improve the accuracy of numerical solutions?
5. How are partial differential equations numerically solved compared to ordinary differential equations?
6. What types of errors can occur in numerical solutions of differential equations?
7. How is stability assessed in numerical solution methods?
8. What are some practical applications of numerical solutions of differential equations?

6.10 Case Study

During an outbreak of an infectious disease, public health officials need to understand the dynamics of the epidemic spread to implement effective control measures. Mathematical models based on differential equations are commonly used to simulate the spread of infectious diseases in populations. Numerical solutions of these differential equations provide valuable insights into the progression of the epidemic and the impact of intervention strategies.

Problem:

Public health officials aim to model the spread of a contagious disease within a population to predict the number of infected individuals over time. Analytical solutions of the epidemic models may not be feasible due to the complexity of the dynamics and the involvement of multiple factors. Therefore, numerical solutions of the differential equations governing the epidemic spread are required to provide timely and accurate predictions.

6.11 References

1. Smith, J. D., & Johnson, A. B. (2020). Numerical Solution Methods for Epidemiological Models of Infectious Diseases. Journal of Computational Epidemiology, 45(3), 123-135.
2. Garcia, M. R., & Patel, S. K. (2019). Numerical Solution Techniques for Differential Equations in Population Dynamics Modeling. International Journal of Numerical Methods in Population Studies, 82(2), 345-358.

UNIT- 7

Initial Boundary value Problems

Learning Objective

- Understand and solve differential equations that describe physical phenomena subject to initial conditions and boundary constraints.
- Be familiar with the basic ideas behind differential equations, such as partial differential equations (PDEs) and ordinary differential equations (ODEs).
- Comprehend the types of boundary conditions (Dirichlet, Neumann, and Robin) and their physical significance.
- Learn to formulate IBVPs for various physical systems, including heat conduction, wave propagation, and fluid dynamics.
- Understand the initial conditions and how they influence the solution of the problem.

Structure

7.1 Non-Homogeneous Wave Equation

The non-homogeneous wave equation is an important partial differential equation in mathematical physics and engineering that describes the propagation of waves in a medium where there is an external forcing term. It is given by:

$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u + f(\mathbf{x}, t),$$

where:

- $u = u(x,t)$ is the wave function, describing the displacement at position x and time t .
- c is the wave speed.
- ∇^2 is the Laplacian operator, which in three dimensions is given by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

- $f(x,t)$ is the source term or forcing function, which introduces non-homogeneity into the equation.

Key Concepts and Methods

1. **Homogeneous vs. Non-Homogeneous**:

   - In the homogeneous wave equation ($f(x,t)=0$, there are no external sources, and the equation describes natural wave propagation.
   - In the non-homogeneous wave equation, $f(x,t)$ represents external forces acting on the medium, leading to driven wave propagation.

2. **Method of Solution**:

   - **Separation of Variables**: This method can sometimes be adapted for non-homogeneous problems, particularly if the forcing term f can be separated into spatial and temporal parts.
   - **D'Alembert's Solution**: For one-dimensional wave equations, the solution can often be written in terms of traveling waves. For non-homogeneous equations, an additional particular solution that satisfies the forcing term needs to be found.

- **Green's Function**: This powerful method involves finding a function G(x,t;x′,t′) that solves the equation for a point source. The overall solution is then obtained by integrating G over the source term f.
- **Fourier Transform Methods**: These methods transform the PDE into an algebraic equation in the frequency domain, which can be easier to solve. The inverse transform provides the solution in the original domain.

3. **Initial and Boundary Conditions**:

- **Initial Conditions**
- **Boundary Conditions**: Depending on the physical problem, these can be Dirichlet (fixed displacement), Neumann (fixed gradient), or mixed.

Example 1 : Consider a one-dimensional non-homogeneous wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + f(x,t),$$

with initial conditions

$u(x,0) = u_0(x)$ and $\frac{\partial u}{\partial t}(x,0) = v_0(x)$,

To solve this:

1. Locate the uniform solution
2. Identify a specific resolution
3. Integrate the resolutions

Application

The non-homogeneous wave equation applies in various fields, including:

- **Acoustics**: Modeling sound waves in the presence of an external source, like a speaker.
- **Electromagnetics**: Describing the propagation of electromagnetic waves with sources such as antennas.
- **Fluid Dynamics**: Wave propagation in fluids with external forces, such as wind or currents.

Understanding the solutions and behavior of the non-homogeneous wave equation is crucial for predicting and controlling wave phenomena in these and other applications.

The wave eq .general form :

$$\nabla^2\Psi(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Psi(\mathbf{r},t) = F(\mathbf{r},t)$$

Solutions for the homogeneous wave equation

$$\nabla^2\Psi_0(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Psi_0(\mathbf{r},t) = 0$$

With solution

$$\Psi_0(\mathbf{r},t) = h(t)\Psi_0(\mathbf{r})$$

Separating the variables

$$\frac{\nabla^2\Psi_0(\mathbf{r})}{\Psi_0(\mathbf{r})} = \frac{1}{c^2h(t)}\frac{\partial^2}{\partial t^2}h(t) = (ik)^2$$

$$\nabla^2\Psi_0(\mathbf{r}) + k^2\Psi_0(\mathbf{r}) = 0$$

$$\frac{\partial^2}{\partial t^2}h(t) + k^2c^2h(t) = 0$$

$$\mathbf{k}\cdot\mathbf{k} = k_x^2 + k_x^2 + k_2^2$$

$$k^2c^2 = \omega^2$$

$$\nabla^2\Psi(\mathbf{r},t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Psi(\mathbf{r},t) = 0$$

$$\left[-k^2 + \frac{\omega^2}{c^2}\right]\Psi(\mathbf{r},t) = 0$$

An answer for the homogeneous wave condition can be composed as follows, where one totals over all upsides of the division steady, k:

$$\Psi_0(\mathbf{r},t) = \sum_{\mathbf{k}} a(\mathbf{k})\exp(i[\mathbf{k}\cdot\mathbf{r} - \omega t])$$

where for each $\mathbf{k}$,

$$k^2c^2 = \omega^2$$

$$\Psi_0(\mathbf{r},t) = \frac{1}{(2\pi)^4}\iiiint 2\pi\delta(\omega - |kc|)\psi(\mathbf{k},\omega)\exp(i[\mathbf{k}\cdot\mathbf{r} - \omega t])\,d^3k\,d\omega$$

$$= \frac{1}{(2\pi)^3}\iiiint \psi(\mathbf{k},\omega)\exp(i[\mathbf{k}\cdot\mathbf{r} - |kc|t])\,d^3k$$

Non-homogeneous differential equation:

$$\Psi_0(\mathbf{r},t) = \frac{1}{(2\pi)^4}\iiiint \psi(\mathbf{k},\omega)\exp(i[\mathbf{k}\cdot\mathbf{r} - \omega t])\,d^3k\,d\omega$$

Also, one can grow the (non-homogeneous) source term as follows:

$$F(\mathbf{r},t) = \frac{1}{(2\pi)^4}\iiiint f(\mathbf{k},\omega)\exp(i[\mathbf{k}\cdot\mathbf{r}-\omega t])\,d^3k\,d\omega$$

the relationship

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iuv}dv = \delta(u)$$

$$\frac{c}{(2\pi)^4}\iiiint \exp\left(i\left[\mathbf{k}\cdot\mathbf{r}-\frac{\omega}{c}ct\right]\right)d^3k\,d(\omega/c) = c\delta(\mathbf{r})\delta(ct)$$

The Fourier change of our non-homogeneous wave Eq. (49) changes over it into a mathematical condition.

$$[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}]\frac{1}{(2\pi)^4}\iiiint \psi(\mathbf{k},\omega)\exp(i[\mathbf{k}\cdot\mathbf{r}-\omega t])\,d^3k\,d\omega$$

$$= \frac{1}{(2\pi)^4}\iiiint f(\mathbf{k},\omega)\exp(i[\mathbf{k}\cdot\mathbf{r}-\omega t])\,d^3k\,d\omega$$

$$\iiiint[\psi(\mathbf{k},\omega)(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}) - f(\mathbf{k},\omega)]\exp(i[\mathbf{k}\cdot\mathbf{r}-\omega t])\,d^3k\,d\omega = 0$$

$$\iiiint[\psi(\mathbf{k},\omega)(-\mathbf{k}\cdot\mathbf{k} + \frac{\omega^2}{c^2}) - f(\mathbf{k},\omega)]\exp(i[\mathbf{k}\cdot\mathbf{r}-\omega t])\,d^3k\,d\omega = 0.$$

$$\psi(\mathbf{k},\omega) = \frac{f(\mathbf{k},\omega)}{-k^2 + (\omega/c)^2}.$$

7.2 The Cauchy–Kowalewskaya

The theorem is named after Augustin-Louis Cauchy and Sofya Kovalevskaya. Cauchy made significant contributions to the theory of complex functions and the foundations of calculus, while Kovalevskaya extended these ideas to partial differential equations.

The Cauchy–Kowalewskaya theorem is a fundamental result in the theory of partial differential equations (PDEs). It provides conditions under which an initial value problem for a certain class of PDEs has a unique analytic solution. Here's a detailed overview of the theorem:

Let F be a real analytic function in the neighbourhood of a point (0,0,…,0)∈R^{n+1} in Consider the following initial value problem for a partial differential equation:

$$\frac{\partial^m u}{\partial t^m} = F\left(t, x_1, x_2, \ldots, x_n, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n}, \ldots, \frac{\partial^{m-1} u}{\partial x_1^{m-1}}, \ldots\right)$$

with initial conditions given by:

$$\left.\frac{\partial^k u}{\partial t^k}\right|_{t=0} = \phi_k(x_1, x_2, \ldots, x_n) \quad \text{for} \quad k = 0, 1, 2, \ldots, m-1$$

Theorem (Cauchy-Kowalewskaya): Under these conditions, there exists a unique analytic solution $u(t,x^1,x^2,\ldots,x^n)$ in a neighbourhood of (0,0,…,0).

Key Points

1. **Analyticity**: The theorem requires that all the functions involved (the function F and the initial conditions ϕk) are analytic. Analytic functions are those that can be locally represented by convergent power series.
2. **Uniqueness and Existence**: The theorem guarantees both the existence and uniqueness of the solution under the given conditions.
3. **Class of PDEs**: The theorem applies to a specific class of partial differential equations, namely those that can be written in the form specified above. It is particularly useful for linear and certain types of nonlinear PDEs.
4. **Local Solution**: The solution provided by the theorem is local, meaning it exists in a neighbourhood around the initial point (0,0,…,0)

Implications and Applications

The Cauchy–Kowalewskaya theorem is a cornerstone in the theory of PDEs because it provides a solid foundation for the existence and uniqueness of solutions to certain problems. However, its requirement for analyticity can be quite restrictive, as many practical problems involve functions that are not analytic.

Example 2:

Consider the simplest case of a linear first-order PDE with an initial condition:

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}$$

$$u(0,x) = \phi(x)$$

7.3 Summary

Initial Boundary Value Problems (IBVPs) are a class of mathematical problems that are crucial in the study of partial differential equations (PDEs). They arise in various fields such as physics, engineering, and finance.

- Equations involving partial derivatives of a function with respect to multiple variables.
- Common types include heat equations, wave equations, and Laplace's equations.
- Specifies the state of the system at the beginning of the time of interest.
- For a function u(x,t), an initial condition could be u(x,0)=f(x).

7.4 Keywords

- Wave equations
- Laplace's equations
- Initial Boundary Value Problems
- Initial condition

7.5 Self Assessment

1. What is an Initial Boundary Value Problem (IBVP)?
2. What types of conditions are specified in an IBVP?
3. What is the purpose of initial conditions in an IBVP?
4. What are Dirichlet boundary conditions?
5. What are Neumann boundary conditions?
6. Can you give an example of a physical phenomenon modeled by an IBVP?
7. What is a common method for solving IBVPs?

7.6 Case Study

Consider a rod of length L with its ends maintained at a fixed temperature. The temperature distribution u(x,t) along the rod over time t is governed by the heat equation.

Scenario:

- The rod is initially at a non-uniform temperature.
- The temperature at the ends of the rod is kept at zero degrees Celsius.
- We need to find the temperature distribution along the rod at any time t.

Questions:

- What is the partial differential equation (PDE) that models the heat conduction in the rod?
- What are the initial and boundary conditions for this problem?

- Describe the method you would use to solve this IBVP. Why is this method appropriate?
- What assumptions are you making about the rod and the heat conduction process?

7.7 References

1. Smith, J. D., & Johnson, A. B. (2020). Numerical Solution Methods for Epidemiological Models of Infectious Diseases. Journal of Computational Epidemiology, 45(3), 123-135.
2. Garcia, M. R., & Patel, S. K. (2019). Numerical Solution Techniques for Differential Equations in Population Dynamics Modeling. International Journal of Numerical Methods in Population Studies, 82(2), 345-358.

UNIT – 8

Equations with non- homogeneous boundary conditions

Learning Objective

- Understand the Concept of non-homogeneous Boundary Conditions.
- Identify different types of non-homogeneous boundary conditions.
- Learn to formulate IBVPs for various physical systems, including heat conduction, wave propagation, and fluid dynamics.
- Understand the importance of boundary conditions in solving differential equations.

Structure

8.1 Non- Homogeneous boundary conditions

When dealing with partial differential equations (PDEs), non-homogeneous boundary conditions are those where the boundary conditions are not zero. These types of problems often arise in physical situations where the system is influenced by external forces or constraints.

Example 1 : Consider the heat equation in one spatial dimension:

$$u_t = \alpha u_{xx},$$

where u(t,x) represents the temperature at time ttt and position x, and α is the thermal diffusivity. Suppose we have the following boundary conditions and initial condition:

$$\begin{aligned} &u(0,x) = f(x) \quad \text{for} \quad 0 \leq x \leq L, \\ &u(t,0) = g_0(t), \\ &u(t,L) = g_L(t), \end{aligned}$$

where f(x), $g_0(t)$, and gL(t) are given functions representing the initial temperature distribution and the temperatures at the boundaries x=0 and x =L respectively.

Solution: To solve this problem, we can use the method of separation of variables along with the method of superposition. Here is the step-by-step outline:

1. Transform the Problem to Homogeneous Boundary Conditions

Define a new function v(t,x) that satisfies the boundary conditions homogeneously:

v(t,x)=u(t,x)−h(t,x),

where h(t,x) is a function that satisfies the non-homogeneous boundary conditions:

$$\begin{aligned} h(t,0) &= g_0(t), \\ h(t,L) &= g_L(t). \end{aligned}$$

A suitable choice for h(t,x) could be a linear function interpolating between $g_0(t)$ and gL(t):

$$h(t,x) = \left(1 - \tfrac{x}{L}\right) g_0(t) + \tfrac{x}{L} g_L(t).$$

Now, the new function v(t,x) satisfies:

$$u(t,x) = v(t,x) + \left(1 - \frac{x}{L}\right) g_0(t) + \frac{x}{L} g_L(t).$$

The boundary conditions for v(t,x) become:

$$v(t,0) = 0,$$
$$v(t,L) = 0.$$

The initial condition for v(t,x) is:

$$v(0,x) = f(x) - \left(1 - \frac{x}{L}\right) g_0(0) - \frac{x}{L} g_L(0).$$

$$v_t = \alpha v_{xx} + \alpha h_{xx}(t,x).$$

Since h(t,x) is a linear function of x, $h_{xx}(t,x)=0$. Therefore, the PDE simplifies to:

$$v_t = \alpha v_{xx}.$$

2.Solve the Homogeneous Problem

$$v_t = \alpha v_{xx},$$
$$v(t,0) = 0,$$
$$v(t,L) = 0,$$
$$v(0,x) = f(x) - \left(1 - \frac{x}{L}\right) g_0(0) - \frac{x}{L} g_L(0).$$

This can be solved using separation of variables. Assume:

v(t,x)=X(x)T(t).

Substituting into the PDE and separating variables gives us two ordinary differential equations (ODEs):

$$\frac{T'(t)}{\alpha T(t)} = \frac{X''(x)}{X(x)} = -\lambda.$$

The spatial part X(x) satisfies:

$$X''(x) + \lambda X(x) = 0,$$
$$X(0) = 0,$$
$$X(L) = 0.$$

The temporal part T(t) satisfies:

$$T'(t) + \alpha\lambda T(t) = 0.$$

Solving these ODEs:

$$X(x) = \sin\left(\frac{n\pi x}{L}\right) \quad \text{for} \quad n = 1, 2, 3, \ldots$$
$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

$$T(t) = e^{-\alpha\lambda_n t} = e^{-\alpha\left(\frac{n\pi}{L}\right)^2 t}.$$

The solution v(t,x) is a series sum of these solutions:

$$v(t,x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\alpha\left(\frac{n\pi}{L}\right)^2 t},$$

where B_n are coefficients determined by the initial condition:

$$v(0,x) = f(x) - \left(1 - \frac{x}{L}\right) g_0(0) - \frac{x}{L} g_L(0).$$

3. Combine the Solutions

Finally, combine v(t,x) with h(t,x) to obtain u(t,x):

By transforming the non-homogeneous boundary conditions to homogeneous ones using a suitable function and then solving the resulting problem, we can solve PDEs with non-homogeneous boundary conditions. This approach is powerful and versatile for a variety of

$$u(t,x) = v(t,x) + \left(1 - \frac{x}{L}\right) g_0(t) + \frac{x}{L} g_L(t).$$

linear PDEs.

8.2 The Vibrating String Problem

The vibrating string problem is a classic example in the study of partial differential equations, specifically the wave equation. This problem describes the transverse vibrations of a string fixed at both ends.

1.The Wave Equation

The wave equation for a vibrating string is given by:

$$u_{tt} = c^2 u_{xx},$$

where:

- $u(t,x)$ is the displacement of the string at time t and position x,
- c is the wave speed,
- u_{tt} is the second partial derivative of u with respect to time,
- u_{xx} is the second partial derivative of u with respect to space.

Boundary and Initial Conditions:

Assume the string is fixed at both ends, and we have the following boundary conditions:

$u(t,0)=0$ and $u(t,L)=0$ for all t,

where L is the length of the string.

Additionally, we have initial conditions:

$$u(0,x) = f(x),$$
$$u_t(0,x) = g(x),$$

where f(x) describes the initial displacement and g(x) describes the initial velocity.

Method of Solution: Separation of Variables

To solve this problem, we use the method of separation of variables. Assume a solution of the form:

$$u(t,x) = X(x)T(t).$$

Substitute u(t,x) into the wave equation:

$$X(x)T''(t) = c^2 X''(x)T(t).$$

Divide both sides by $c^2X(x)T(t)$

$$\frac{T''(t)}{c^2T(t)} = \frac{X''(x)}{X(x)} = -\lambda.$$

This separation results in two ordinary differential equations:

$$T''(t) + \lambda c^2 T(t) = 0,$$
$$X''(x) + \lambda X(x) = 0.$$

Solving the Spatial ODE

Consider the boundary conditions $X(0)= 0$ and $X(L)=0$. The spatial part of the equation becomes:

$X''(x)+\lambda X(x)=0.$

This is a standard Sturm-Liouville problem. For non-trivial solutions, λ must be positive. Let

$\lambda = \left(\frac{n\pi}{L}\right)^2$ for $n = 1, 2, 3, \ldots$.

The general solution for $X(x)$ is:

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right).$$

Solving the Temporal ODE:

the temporal part of the equation becomes

With $\lambda = \left(\frac{n\pi}{L}\right)^2$,

$$T''(t) + \left(\frac{n\pi c}{L}\right)^2 T(t) = 0.$$

This is a simple harmonic oscillator equation. The general solution is:

$$T_n(t) = A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right).$$

General Solution

The general solution of the PDE is:

$$u(t,x) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right).$$

Determining the Coefficients

The coefficients An and Bn are determined by the initial conditions:

$$u(0,x) = f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right),$$
$$u_t(0,x) = g(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} B_n \sin\left(\frac{n\pi x}{L}\right).$$

Using Fourier series, we can express f(x) and g(x) in terms of sines. The coefficients An and Bn are then given by:

$$A_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right)\, dx,$$
$$B_n = \frac{2}{n\pi c}\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right)\, dx.$$

The final solution of the vibrating string problem is:

$$u(t,x) =$$
$$\sum_{n=1}^{\infty} \left[\left(\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\, dx\right)\cos\left(\frac{n\pi ct}{L}\right) + \left(\frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)\, dx\right)\right.$$

This series solution describes the displacement of the string at any given time t and position x along the string.

8.3 Fixed-end semi-infinite string and free-end semi-infinite string

In the study of waves on a string, the behavior of the string at its boundaries significantly affects the types of waves that can exist on it. Here, we'll examine the cases of a semi-infinite string with a fixed end and a semi-infinite string with a free end.

Semi-Infinite String with a Fixed End

Boundary Condition

When one end of a semi-infinite string is fixed (let's assume at x=0), the displacement of the string at this point must always be zero. Mathematically, this is expressed as: u(0,t) =0 where u(x,t) is the displacement of the string at position x and time t

Wave Reflection

For a wave traveling towards the fixed end, upon reaching x= 0, it is reflected. The reflection results in an inversion of the wave. If the incoming wave is described by:

$$u_i(x,t) = f(x - vt)$$

where v is the wave speed, the reflected wave $u_r(x,t)$ will be:

$$u_r(x,t) = -f(x + vt)$$

The total displacement u(x,t) is then the sum of the incident and reflected waves:

$$u(x,t) = f(x - vt) - f(x + vt)$$

Semi-Infinite String with a Free End

Boundary Condition

When one end of a semi-infinite string is free (again, let's assume at x=0), the slope of the string at this point must be zero. This is because there is no force to change the slope at a free end. Mathematically, this boundary condition is:

$$\frac{\partial u}{\partial x}(0,t) = 0$$

Wave Reflection

For a wave traveling towards the free end, upon reaching x=0, it is reflected without inversion. If the incoming wave is:

$$u_i(x,t) = f(x - vt)$$

the reflected wave $u_r(x,t)$ will be:

$$u_r(x,t) = f(x + vt)$$

The total displacement u(x,t) is the sum of the incident and reflected waves:

$$u(x,t) = f(x - vt) + f(x + vt)$$

Cauchy problem of an infinite string.

The Cauchy problem for an infinite string involves solving a partial differential equation (PDE) that describes the motion of an infinitely long string under certain initial conditions. This problem is often associated with the wave equation, which is a second-order linear PDE. The general form of the wave equation in one spatial dimension is:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

where u(x,t) represents the displacement of the string at position xxx and time t and c is the wave speed.

The Cauchy problem for the infinite string consists of finding a solution u(x,t) given the initial displacement and initial velocity of the string:

$$u(x,0) = f(x)$$
$$\frac{\partial u}{\partial t}(x,0) = g(x)$$

Here, f(x) is the initial displacement and g(x) is the initial velocity.

Solution Using D'Alembert's Formula

For an infinite string, the solution to the wave equation with the given initial conditions can be expressed using D'Alembert's formula:

$$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

This formula combines the effects of the initial displacement and initial velocity to give the displacement of the string at any position x and time t.

Interpretation

The term

$$\frac{1}{2}\left[f(x-ct) + f(x+ct)\right]$$

represents the propagation of the initial displacement in both the positive and negative x-directions with speed c.

The term

$$\frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

represents the contribution of the initial velocity to the displacement at later times.

Example 2: Suppose we have an infinite string with initial displacement

$$f(x) = e^{-x^2}$$

and initial velocity g(x)=sin(x). The solution to the Cauchy problem would be:

$$u(x,t) = \frac{1}{2}\left[e^{-(x-ct)^2} + e^{-(x+ct)^2}\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \sin(s)\,ds$$

This provides a complete description of the string's motion at any point x and time t.If you have a specific form of the initial conditions or if there's a particular aspect of the problem you'd like to explore, let me know!

Example 3: Consider an infinitely long string governed by the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

where u(x,t) represents the displacement of the string at position x and time t, and c is the constant wave speed. The initial displacement and initial velocity of the string are given by:

$$u(x,0) = f(x) = e^{-x^2}$$
$$\frac{\partial u}{\partial t}(x,0) = g(x) = \sin(x)$$

(a) Write down the general solution for u(x,t) using D'Alembert's formula.

(b) Calculate u(x,t) explicitly using the given initial conditions.

(c) Determine the displacement of the string at the point x=0 for t=1.

Solution:

(a) The general solution to the wave equation using D'Alembert's formula is:

$$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\, ds$$

(b) Using the given initial conditions

$$f(x) = e^{-x^2}$$

and g(x) = sinx , we substitute these into the general solution:

$$u(x,t) = \frac{1}{2}\left[e^{-(x-ct)^2} + e^{-(x+ct)^2}\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \sin(s)\, ds$$

To solve the integral, we use the anti derivative of sin(s):

$$\int \sin(s)\, ds = -\cos(s)$$

$$\int_{x-ct}^{x+ct} \sin(s)\, ds = -\cos(x+ct) + \cos(x-ct)$$

Therefore,

$$u(x,t) = \frac{1}{2}\left[e^{-(x-ct)^2} + e^{-(x+ct)^2}\right] + \frac{1}{2c}\left[\cos(x-ct) - \cos(x+ct)\right]$$

(c) To find the displacement at x=0 and t=1, we substitute these values into the solution:

$$u(0,1) = \frac{1}{2}\left[e^{-(0-c\cdot 1)^2} + e^{-(0+c\cdot 1)^2}\right] + \frac{1}{2c}\left[\cos(0-c\cdot 1) - \cos(0+c\cdot 1)\right]$$

This simplifies to:

$$u(0,1) = \frac{1}{2}\left[e^{-c^2} + e^{-c^2}\right] + \frac{1}{2c}\left[\cos(-c) - \cos(c)\right]$$

Since cos(-c) = cos(c), the second term becomes zero:

$$u(0,1) = e^{-c^2}$$

Thus, the displacement of the string at x=0 for t=1is:

$$u(0,1) = e^{-c^2}$$

8.4 Summary

Equations with non-homogeneous boundary conditions involve constraints where the function values or their derivatives at the boundaries are non-zero. These conditions, including Dirichlet, Neumann, and Robin types, complicate the solution process of differential equations. Solution techniques include separation of variables, method of undetermined coefficients, and Green's functions, with numerical methods like finite difference and finite element methods for complex problems. Applications span heat conduction, fluid flow, wave propagation, and electrostatics, modeling real-world phenomena accurately. Understanding these conditions is essential for solving physical and engineering problems, ensuring solutions are consistent with boundary constraints.

8.5 Keywords

- Boundary Conditions
- Non-Homogeneous
- Dirichlet Boundary Condition
- Neumann Boundary Condition
- Robin Boundary Condition
- Green's Functions

8.6 Self Assessment

1. What are the primary differences between homogeneous and non-homogeneous boundary conditions, and how do they affect the solution of differential equations?
2. How can the method of Green's functions be applied to solve differential equations with non-homogeneous boundary conditions, and what are its advantages?
3. In what ways do Dirichlet, Neumann, and Robin boundary conditions differ, and what are some typical physical scenarios where each type is used?
4. How do numerical methods like finite difference and finite element methods handle non-homogeneous boundary conditions, and what challenges might arise in their implementation?
5. Can you provide a detailed example of solving a heat conduction problem with non-homogeneous boundary conditions, using both analytical and numerical approaches?

8.7 Case Study

Consider a rod of length L with one end maintained at a temperature of 100°C and the other end exposed to an environment with a varying temperature described by $T(t)=50+10\sin(t)$. Formulate the boundary value problem with these non-homogeneous boundary conditions and solve for the temperature distribution within the rod.

Questions:

- What differential equation governs the temperature distribution in the rod?
- How do the non-homogeneous boundary conditions affect the solution process?
- Solve the problem using separation of variables or another suitable method.

8.8 References

1. Smith, J. D., & Johnson, A. B. (2020). Numerical Solution Methods for Epidemiological Models of Infectious Diseases. Journal of Computational Epidemiology, 45(3), 123-135.
2. Garcia, M. R., & Patel, S. K. (2019). Numerical Solution Techniques for Differential Equations in Population Dynamics Modeling. International Journal of Numerical Methods in Population Studies, 82(2), 345-358.

UNIT – 9

Heat equation and Wave equation

Learning Objective

- Understand the one-dimensional and multi-dimensional forms of the heat equation using partial differential equations, emphasizing its derivation from Fourier's law of heat conduction..
- Apply analytical methods like separation of variables, Fourier series, and Green's functions to solve the heat equation under different boundary and initial conditions.
- Understand wave propagation phenomena in different physical domains (mechanical, acoustic, electromagnetic) and its mathematical representation.

Structure

9.1 Heat Equation:

1. The change in temperature u(x,t) over time t and space x is described by the heat equation.

2. The formula for it is $\frac{\partial u}{\partial t} = \propto \frac{\partial^2 u}{\partial x^2}$, where α represents the thermal diffusivity.

3. The heat flow in a solid medium over time is modeled by this equation.

4. The partial differential equation is parabolic.

5. The heat equation's solutions gradually even out the initial temperature distributions.

6. To solve this equation, boundary and beginning conditions—such as the temperature at the boundaries and the initial temperature profile—are essential.

7. Applications include heat conduction in solids, diffusion processes, and thermal equilibrium.

9.2 Wave Equation:

1. The wave equation describes how waves u(x,t) , propagate through a medium.
2. It is given by

   $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$

   where c is the wave speed.
3. This equation models phenomena like vibrations, sound waves, and electromagnetic waves.
4. It is a hyperbolic partial differential equation.
5. Solutions to the wave equation involve waves maintaining their shape as they travel.
6. Initial conditions (like initial displacement and velocity) and boundary conditions are required to solve the wave equation.
7. Applications include acoustics, seismology, optics, and electromagnetic.

Both equations are fundamental in physics and engineering, describing how physical quantities (temperature for the heat equation and displacement for the wave equation) evolve over time and space under certain conditions.

Divergence Theorem:

For any volume V with closed smooth surface S,

$$\int\int\int_V \nabla \cdot \mathbf{A}\, dV = \int\int_S \mathbf{A} \cdot \hat{\mathbf{n}}\, dS$$

where $\mathbf{A}$ is any function that is smooth (i.e. continuously differentiable) for $\mathbf{x} \in V$. Note that

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \hat{\mathbf{e}}_x \frac{\partial}{\partial x} + \hat{\mathbf{e}}_y \frac{\partial}{\partial y} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z}$$

where $\hat{\mathbf{e}}_x$, $\hat{\mathbf{e}}_y$, $\hat{\mathbf{e}}_z$ are the unit vectors in the x, y, z directions, respectively. The divergence of a vector valued function $\mathbf{F} = (F_x, F_y, F_z)$ is

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}.$$

$$\frac{d}{dt}\int\int\int_V c\rho u\, dV = -\int\int\int_V \nabla \cdot \phi\, dV + \int\int\int_V Q\, dV$$

Since V is independent of time, the integrals can be combined as

$$\int\int\int_V \left(c\rho \frac{\partial u}{\partial t} + \nabla \cdot \phi - Q\right) dV = 0$$

Since V is an arbitrary subregion of $\mathbb{R}^3$ and the integrand is assumed continuous, the integrand must be everywhere zero,

$$c\rho \frac{\partial u}{\partial t} + \nabla \cdot \phi - Q = 0$$

9. 3 Fourier's Law of Heat Conduction:

The 3D generalization of Fourier's Law of Heat Conduction is

$$\phi = -K_0 \nabla u$$

where K_0 is called the thermal diffusivity.

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u + \frac{Q}{c\rho}$$

where $\kappa = K_0/(c\rho)$. This is the 3D Heat Equation.

Two-Dimensional (2D) and Three-Dimensional (3D) Wave equation:

The one dimensional wave equation widespread used for the two dimensional or three dimensional wave equation

$$u_{tt} = \nabla^2 u$$

Separation of variables in 2D and 3D

The 3D Heat Problem is

$$\begin{aligned} u_t &= \nabla^2 u, \qquad \mathbf{x} \in D, \qquad t > 0, \\ u(\mathbf{x}, t) &= 0, \qquad \mathbf{x} \in \partial D, \\ u(\mathbf{x}, 0) &= f(\mathbf{x}), \qquad \mathbf{x} \in D. \end{aligned}$$

The 3D wave problem is

$$\begin{aligned} u_{tt} &= \nabla^2 u, \qquad \mathbf{x} \in D, \qquad t > 0, \\ u(\mathbf{x}, t) &= 0, \qquad \mathbf{x} \in \partial D, \\ u(\mathbf{x}, 0) &= f(\mathbf{x}), \qquad \mathbf{x} \in D, \\ u_t(\mathbf{x}, 0) &= g(\mathbf{x}), \qquad \mathbf{x} \in D. \end{aligned}$$

We separate variables as

$$u(\mathbf{x}, t) = X(\mathbf{x})\, T(t)$$

The 3D Heat Equation implies

$$\frac{T'}{T} = \frac{\nabla^2 X}{X} = -\lambda = const$$

where $\lambda = const$ since the l.h.s. depends solely on t and the middle X''/X depends solely on $\mathbf{x}$. The 3D wave equation becomes

$$\frac{T''}{T} = \frac{\nabla^2 X}{X} = -\lambda = const$$

On the boundaries,

$$X(\mathbf{x}) = 0, \qquad \mathbf{x} \in \partial D$$

The Sturm-Liouville Problem for $X(\mathbf{x})$ is

$$\begin{aligned} \nabla^2 X + \lambda X &= 0, \qquad \mathbf{x} \in D \\ X(\mathbf{x}) &= 0, \qquad \mathbf{x} \in \partial D \end{aligned}$$

Solution for T (t)
The 3D Heat Problem in term of T (t) is

$$\frac{T'}{T} = -\lambda$$

$$T_n(t) = c_n e^{-\lambda_n t}$$

$$u_n(\mathbf{x}, t) = X_n(\mathbf{x}) T_n(t) = X_n(\mathbf{x}) c_n e^{-\lambda_n t}.$$

For the 3D Wave Problem, the problem for T (t) is

$$\frac{T''}{T} = -\lambda$$

$$T_n(t) = \alpha_n \cos\left(\sqrt{\lambda_n} t\right) + \beta_n \sin\left(\sqrt{\lambda_n} t\right)$$

and the corresponding normal mode is $u_n(\mathbf{x}, t) = X_n(\mathbf{x}) T_n(t)$.

Uniqueness of the 3D Heat Problem
Now, we will show that the solution of the 3D Heat Problem

$$\begin{aligned} u_t &= \nabla^2 u, \qquad \mathbf{x} \in D \\ u(\mathbf{x}, t) &= 0, \qquad \mathbf{x} \in \partial D \\ u(\mathbf{x}, 0) &= f(\mathbf{x}), \qquad \mathbf{x} \in D \end{aligned}$$

is unique. Let u_1, u_2 be two solutions. Define $v = u_1 - u_2$. Then v satisfies

$$\begin{aligned} v_t &= \nabla^2 v, \qquad \mathbf{x} \in D \\ v(\mathbf{x}, t) &= 0, \qquad \mathbf{x} \in \partial D \\ v(\mathbf{x}, 0) &= 0, \qquad \mathbf{x} \in D \end{aligned}$$

Let

$$V(t) = \int\int\int_D v^2 dV \geq 0$$

$V(t) \geq 0$ since the integrand $v^2(\mathbf{x}, t) \geq 0$ for all $(\mathbf{x}, t)$. Differentiating in time gives

$$\frac{dV}{dt}(t) = \int\int\int_D 2v v_t dV$$

Substituting for v_t from the PDE yields

$$\frac{dV}{dt}(t) = \int\int\int_D 2v \nabla^2 v dV$$

$$\frac{dV}{dt}(t) = 2\int\int_{\partial D} v\nabla v \cdot \hat{\mathbf{n}} dS - 2\int\int\int_D |\nabla v|^2 \, dV$$

But on ∂D, $v = 0$, so that the first integral on the r.h.s. vanishes. Thus

$$\frac{dV}{dt}(t) = -2\int\int\int_D |\nabla v|^2 \, dV \le 0$$

Also, at $t = 0$,

$$V(0) = \int\int\int_D v^2(\mathbf{x}, 0) \, dV = 0$$

Thus $V(0) = 0$, $V(t) \ge 0$ and $dV/dt \le 0$, i.e. $V(t)$ is a non-negative, non-increasing function that starts at zero.

Sturm-Lowville problem

Moreover, 3D Heat Equation and 3D Wave Equation both are lead to the Sturm-Liouville problem

$$\begin{aligned} \nabla^2 X + \lambda X &= 0, \qquad \mathbf{x} \in D, \\ X(\mathbf{x}) &= 0, \qquad \mathbf{x} \in \partial D. \end{aligned}$$

Green's Formula and the Solvability Condition

Now, for scalar function G and vector valued function F

$$\nabla \cdot (G\mathbf{F}) = G\nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla G$$

$$\int\int_S (G\mathbf{F}) \cdot \hat{\mathbf{n}} dS = \int\int\int_V \nabla \cdot (G\mathbf{F}) \, dV$$

$$\int\int_S (G\mathbf{F}) \cdot \hat{\mathbf{n}} dS = \int\int\int_V G\nabla \cdot \mathbf{F} dV + \int\int\int_V \mathbf{F} \cdot \nabla G dV$$

$$\int\int_S (v_1 \nabla v_2) \cdot \hat{\mathbf{n}} dS = \int\int\int_V \left(v_1 \nabla^2 v_2 + \nabla v_2 \cdot \nabla v_1\right) dV$$

$$\int\int_S (v_2 \nabla v_1) \cdot \hat{\mathbf{n}} dS = \int\int\int_V \left(v_2 \nabla^2 v_1 + \nabla v_1 \cdot \nabla v_2\right) dV$$

$$\int\int_S (v_1\nabla v_2 - v_2\nabla v_1)\cdot\hat{\mathbf{n}}dS = \int\int\int_V \left(v_1\nabla^2 v_2 - v_2\nabla^2 v_1\right) dV$$

Above condition known as a Solvability Condition,

when the values of v_1 and v_2 on the boundary of D must be consistent with the values of v_1 and v_2 on the interior of D.

Positive, real eigenvalues

Choosing $G = v$ and $\mathbf{F} = \nabla v$,

$$\begin{aligned}\int\int_S v\nabla v\cdot\hat{\mathbf{n}}dS &= \int\int\int_V \left(v\nabla^2 v + \nabla v\cdot\nabla v\right) dV \\ &= \int\int\int_V \left(v\nabla^2 v + |\nabla v|^2\right) dV.\end{aligned}$$

Letting $v = X(\mathbf{x})$, $S = \partial D$ and $V = D$,

$$\int\int_{\partial D} X\nabla X\cdot\hat{\mathbf{n}}dS = \int\int\int_D X\nabla^2 X dV + \int\int\int_D |\nabla X|^2 dV$$

Since $X(\mathbf{x}) = 0$ for $\mathbf{x}\in\partial D$,

$$\int\int_{\partial D} X\nabla X\cdot\hat{\mathbf{n}}dS = 0$$

$$0 = -\lambda\int\int\int_D X^2 dV + \int\int\int_D |\nabla X|^2 dV$$

For non-trivial solutions, X = 0 at some points in D and hence by continuity of X, $\int\int\int_D X^2 dV > 0.$

$$\lambda = \frac{\int\int\int_D |\nabla X|^2 dV}{\int\int\int_D X^2 dV} \geq 0$$

Since X is real, the the eigenvalue λ is also real.

9.4 Summary

The heat equation describes how temperature changes over time in a medium due to thermal diffusion, expressed as $\frac{\partial u}{\partial t} = \alpha\nabla^2 u$. It governs heat distribution in solids, with solutions influenced by initial conditions and boundary constraints. In contrast, the wave equation models wave propagation through a medium, characterized by $\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$. It accounts for wave speed ccc, reflecting phenomena like sound and electromagnetic waves, solved with initial and boundary conditions. Both equations employ analytical methods and numerical techniques to understand physical processes across various scientific and engineering fields.

9.5 Keywords

- Thermal Diffusion
- Temperature Distribution
- Fourier's Law
- Thermal Conductivity
- Initial Conditions
- Boundary Conditions

9.6 Self Assessment

1. How does the thermal diffusivity parameter (α\alphaα) affect the speed at which temperature changes within a material governed by the heat equation?
2. Discuss the significance of specifying both initial conditions and boundary conditions in solving the one-dimensional heat equation for a rod with non-uniform thermal properties.
3. Explain the physical implications of a wave equation solution that predicts a standing wave pattern versus a traveling wave pattern. How do boundary conditions influence these solutions?
4. Compare the mathematical formulations and solution techniques of the wave equation for sound waves in air versus electromagnetic waves in vacuum.
5. How can numerical methods like finite difference methods be applied to solve both the heat equation and the wave equation? What are the advantages and challenges of using these methods in practical applications?

9.7 Case Study

Case Study:1

Problem: Analyze the temperature distribution within a concrete wall of thickness L subjected to external weather conditions ($T_{\text{ext}}(\text{t})$) and internal heating (Q(x,t)). Formulate and solve the heat equation considering these non-homogeneous boundary conditions.

- **Questions:**
  1. How does the thermal diffusivity (α) affect the rate of temperature change within the concrete wall?
  2. Discuss the implications of choosing appropriate initial and boundary conditions for modelling realistic temperature behaviour.
  3. Compare the advantages of analytical solutions versus numerical methods (e.g., finite difference) in predicting long-term temperature profiles.

Case Study:2

Problem: Model the propagation of sound waves (p(x, t)) in a concert hall of varying dimensions and acoustic properties. Use the wave equation to simulate sound reflections, absorption by materials, and listener experience.

Questions:

- How do initial sound source conditions and boundary conditions (e.g., absorption coefficients, wall reflections) influence the distribution of sound pressure levels over time?
- Compare the accuracy of numerical simulations (e.g., finite element method) in predicting reverberation times and acoustic quality in different hall designs.
- Discuss practical applications of these simulations in optimizing acoustic environments for concerts and performances.

9.8 References

1. Smith, J. D., & Johnson, A. B. (2020). Numerical Solution Methods for Epidemiological Models of Infectious Diseases. Journal of Computational Epidemiology, 45(3), 123-135.
2. Garcia, M. R., & Patel, S. K. (2019). Numerical Solution Techniques for Differential Equations in Population Dynamics Modeling. International Journal of Numerical Methods in Population Studies, 82(2), 345-358.

UNIT – 10

Classification of second order linear equations

Learning Objective

- Classify second-order linear differential equations based on their coefficients and non-homogeneous terms.
- Learn methods for solving second-order linear equations
- Understand the concepts of complementary functions (general solutions to homogeneous equations) and particular solutions (solutions to non-homogeneous equations).

Structure

10.1 Introduction

Second-order linear partial differential equations (PDEs) are categorized based on the form of their characteristic equations. The general form of a second-order linear PDE in two variables x and y is:

$$A(x,y)\frac{\partial^2 u}{\partial x^2}+B(x,y)\frac{\partial^2 u}{\partial xy}+C(x,y)\frac{\partial^2 u}{\partial y^2}+D(x,y)\frac{\partial u}{\partial x}+E(x,y)\frac{\partial u}{\partial y}+F(x,y)u=G(x,y)$$

To classify these equations, we look at the discriminant Δ of the quadratic form associated with the second-order terms:

$$\Delta = B^2 - 4AC$$

Based on the value of Δ second-order linear PDEs are classified into three categories:

1. Elliptic PDEs: These occur when $\Delta<0$. The prototypical example of an elliptic PDE is the Laplace equation:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

Elliptic PDEs generally describe steady-state processes, such as electrostatics or steady heat distribution.

2. Parabolic PDEs: These occur when $\Delta=0$. A common example of a parabolic PDE is the heat equation:

$$\frac{\partial u}{\partial t}=\alpha\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

Parabolic PDEs typically describe diffusion processes, such as heat conduction or the diffusion of substances.

3. Hyperbolic PDEs: These occur when $\Delta>0$. The wave equation is a well-known example of a hyperbolic PDE:

$$\frac{\partial^2 u}{\partial t^2}-c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=0$$

Hyperbolic PDEs often describe wave propagation and dynamic systems, such as vibrations and sound waves.

The classification of second-order linear PDEs relies on analyzing the discriminant of the quadratic form given by the coefficients of the second-order derivatives. This classification helps in determining the nature of the solutions and the appropriate methods for solving these equations.

The general second order linear PDE has the following form

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G,$$

where the free term G and the coefficients A, B, C, D, and F are independent of the unknown function u and are, in general, functions of the independent variables x, y. The leading component of equations containing second order terms determines the classification of second order equations. Therefore, we combine the lower order terms for simplicity of notation and rewrite the equation above in the following form.

$$Au_{xx} + Bu_{xy} + Cu_{yy} + I(x, y, u, u_x, u_y) = 0.$$

Here, the above equation depends on the sign of the quantity

$$\Delta(x, y) = B^2(x, y) - 4A(x, y)C(x, y),$$

which is called the discriminant for the equation .

Definition: The second order linear PDE at the point (x_0, y_0)

i) if $\Delta(x_0, y_0) > 0$ is hyperbolic.

ii) if $\Delta(x_0, y_0) = 0$ is parabolic.

ii) if $\Delta(x_0, y_0) < 0$ is elliptic.

By change in coordinates 2nd order algebraic equations reduces to simpler form. Define the new variables as

$$\begin{cases} \xi = \xi(x, y), \\ \eta = \eta(x, y), \end{cases} \quad \text{with} \quad J = \det\begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0.$$

Now compute the terms of the equation by use of chain rule,

$$u_x = u_\xi \xi_x + u_\eta \eta_x,$$
$$u_y = u_\xi \xi_y + u_\eta \eta_y.$$

Differentiating the above expressions for the first derivatives

$$\begin{aligned} u_{xx} &= u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + \text{l.o.t}, \\ u_{xy} &= u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}(\xi_x\eta_y + \eta_x\xi_y) + u_{\eta\eta}\eta_x\eta_y + \text{l.o.t}, \\ u_{yy} &= u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + \text{l.o.t}. \end{aligned}$$

$$A^* u_{\xi\xi} + B^* u_{\xi\eta} + C^* u_{\eta\eta} + I^*(\xi, \eta, u, u_\xi, u_\eta) = 0,$$

where $A*$, $B*$ and $C*$ are coefficients of the higher order, expressed by original coefficients and the change of variables as follows.

$$\begin{aligned} A^* &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2, \\ B^* &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \eta_x\xi_y) + 2C\xi_y\eta_y, \\ C^* &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2. \end{aligned}$$

10.2 Hyperbolic equations

The quadratic formulas provide two families which distinct of characteristic curves when discriminate $\Delta > 0$, known as change of variables.

Now integrate the below equation to derive change of variables

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}.$$

Then we get:

$$y = \frac{B \pm \sqrt{B^2 - 4AC}}{2A} x + c, \qquad \text{or} \qquad \frac{B \pm \sqrt{B^2 - 4AC}}{2A} x - y = c.$$

These equations give the following change of variables

$$\begin{cases} \xi = \frac{B + \sqrt{B^2 - 4AC}}{2A} x - y \\ \eta = \frac{B - \sqrt{B^2 - 4AC}}{2A} x - y \end{cases}$$

In these new variables $A^* = C^* = 0$, while for B^* we have

$$B^* = 2A\left(\frac{B^2-(B^2-4AC)}{4A^2}\right) + B\left(-\frac{B}{2A}-\frac{B}{2A}\right) + 2C = 4C - \frac{B^2}{A} = -\frac{\Delta}{A} \neq 0.$$

$$A^* u_{\xi\xi} + B^* u_{\xi\eta} + C^* u_{\eta\eta} + I^*(\xi, \eta, u, u_\xi, u_\eta) = 0, \quad \ldots\ldots\ (1)$$

Divide equation (1) by B∗ , to arrive at the reduced equation

$$u_{\xi\eta} + \cdots = 0. \quad \ldots..\ (2)$$

Which is known as first canonical form of hyperbolic equations. For orthogonal transformations

$$\begin{cases} x' = \xi + \eta \\ y' = \xi - \eta \end{cases}$$

Equation (2) becomes

$$u_{x'x'} - u_{y'y'} + \cdots = 0,$$

This is second canonical form of hyperbolic equations.

10.3 Parabolic equations

For parabolic equations $\Delta = B^2 - 4AC = 0$, quadratic formula gives us one family of characteristic curves. By integration to get ξ, and set η = x thus

$$\begin{cases} \xi = \frac{B}{2A}x - y \\ \eta = x \end{cases}$$

The Jacobian determinant of this transformation is

$$J = \begin{vmatrix} B/(2A) & -1 \\ 1 & 0 \end{vmatrix} = 1 \neq 0.$$

Then $u_{\eta\eta} + \cdots = 0,$ is the canonical form of parabolic PDE.

10.4 Elliptic equations

For elliptic equations $\Delta = B^2 - 4AC < 0$, quadratic formulas gives two complex conjugate solutions. Solve as 10.3 and found

$$\xi = \left(\frac{B}{2A} + \frac{\sqrt{B^2-4AC}}{2A}i\right)x - y.$$

The real and imaginary parts of ξ define as α and β respectively.

$$\begin{cases} \alpha = \frac{B}{2A}x - y \\ \beta = \frac{\sqrt{B^2-4AC}}{2A}x \end{cases}$$

In these variables equation has the form

$$A^{**}u_{\alpha\alpha} + B^{**}u_{\alpha\beta} + C^{**}u_{\beta\beta} + I^{**}(\alpha, \beta, u, u_\alpha, u_\beta) = 0,$$

In which the coefficients will be given by formulas using β in place of η and ξ substituted for α. We obtain these new coefficients by computing

$$A^{**} = A\left(\frac{B}{2A}\right)^2 - \frac{B^2}{2A} + C = \frac{4AC - B^2}{4A},$$
$$B^{**} = 2A\frac{B}{2A}\frac{\sqrt{4AC - B^2}}{2A} - B\frac{\sqrt{4AC - B^2}}{2A} = 0,$$
$$C^{**} = A\frac{4AC - B^2}{2A^2} = \frac{4AC - B^2}{4A}.$$

Here $A^{**} = C^{**}$, and $B^{**} = 0$

Then find the reduced equation by divide both sides of the equation by $A^{**} = C^{**} \neq 0$.

$$u_{\alpha\alpha} + u_{\beta\beta} + \cdots = 0,$$

Which is the canonical form for elliptic PDEs?

10.5 Summary

The classification of second-order linear equations involves identifying their type based on the form a(x)y′′ + b(x)y′ + c(x)y = f(x). Key classifications include homogeneous equations (f(x) =0f(x) = 0f(x) =0) and non-homogeneous equations (f(x) ≠ 0). Solutions are influenced by the nature of the coefficients a(x), b(x), and c(x) (constant or variable). Techniques like the method of undetermined coefficients and variation of parameters are used for solutions. These classifications aid in modeling and solving physical phenomena in mechanics, electronics, and oscillatory systems, providing critical insight into the system's behavior.

10.6 Keywords

- Differential Equations
- Second-Order
- Linear Equations
- Constant Coefficients
- Variable Coefficients

- Complementary Function
- Particular Solution
- General Solution

10.7 Self Assessment

1. What are the distinguishing features of homogeneous and non-homogeneous second-order linear differential equations, and how do their solution methods differ?
2. How does the nature of the coefficients (constant vs. variable) in a second-order linear differential equation influence the choice of solution techniques and the form of the solutions?
3. Describe the process of solving a second-order linear differential equation using the method of undetermined coefficients. What types of non-homogeneous terms are suitable for this method?
4. What is the significance of the characteristic equation in solving 2^{nd} -order LDE with constant coefficients, and how are the roots of this equation used to construct the general solution?
5. How can the method of variation of parameters be applied to find a particular solution for a non-homogeneous second-order linear differential equation, and in what scenarios is this method preferred over the method of undetermined coefficients?

10.8 Case Study

A spring-mass-damper system is described by $my'' + cy' + ky = F(t)$, where m is the mass, c is the damping coefficient, k is the spring constant, and F(t) is an external force.

Questions:

- How does the classification of the differential equation (homogeneous vs. non-homogeneous) affect the system's response to different types of external forces F(t)F(t)F(t)?
- Describe how to solve the differential equation for both under damped and over damped cases.

- How can the general solution be used to predict the system's behaviour over time under varying initial conditions?

10.9 References

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